Combinatorics, external rays, and twisted polynomials.

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1. Background.
2. The Stripping Algorithm.
3. Early returns.
4. Non-admissible combinatorics.
5. The proof.
6. Recapture of the critical point.

The images were made with Mandel, a program available from www.mndynamics.com.

Talk given at the conference on dynamical systems in Göttingen, August 2011.
Iteration of $f_c(z) = z^2 + c$.

Filled Julia set $\mathcal{K}_c = \{ z \in \mathbb{C} \mid f_c^n(z) \not\to \infty \}$. 

Mandelbrot set $\mathcal{M} = \{ c \in \mathbb{C} \mid c \in \mathcal{K}_c \}$. 

Main tool: External rays with rational angles. Defined by conformal mappings $\Phi_c : \hat{\mathbb{C}} \setminus \mathcal{K}_c \to \hat{\mathbb{C}} \setminus \hat{\mathbb{D}}$ and $\Phi_M : \hat{\mathbb{C}} \setminus \mathcal{M} \to \hat{\mathbb{C}} \setminus \hat{\mathbb{D}}$.

Applications include: description of points and subsets; topological models.

Other combinatorial concepts: Hubbard trees, kneading sequences, and internal addresses.
1.1 External rays

Polynomial $f_c(z)$ corresponds to angle doubling modulo 1.
Symbolic dynamics of binary digits.

Landing of dynamic rays with rational angle $\theta$:
Odd denominator — periodic digits — landing at periodic point.
Even denominator — preperiodic digits — landing at preperiodic point.

Landing pattern is stable except for bifurcations; Mandel demo 3, p. 5, 6, 10.
Parameter rays are landing at roots and Misiurewicz points.
1.2 Hubbard tree

Postcritically finite polynomial \( f_c \); here 5-periodic.

Concrete Hubbard tree: connect postcritical points (and 0) by arcs within \( \mathcal{K}_c \).

Oriented Hubbard tree: considered as a planar graph.

There are two components \( A \) and \( B \) at the critical point \( z = 0 \).

Equivalently: the critical value \( z = c \) is an endpoint.
1.3 Kneading sequence

Symbolic dynamics of sequence $AABA^*$. 

$j$-th symbol says whether $f_{c}^{j-1}(c)$ is in $A$, in $B$, or $= 0$. 

Or for angle $\theta$: $A$ corresponds to $2^j\theta \in \left(\frac{\theta}{2}, \frac{\theta+1}{2}\right)$ . . . 

Both characteristic angles, here $\theta_- = \frac{5}{31}$ and $\theta_+ = \frac{6}{31}$, give the same kneading sequence: the yellow strip is forbidden.
1.4 Internal address

The internal address is describing a kneading sequence by increasing periods. These correspond to hyperbolic components in $\mathcal{M}$, where the kneading sequence is changing.

Example: $\overline{AABA*}$ is obtained by changing $\overline{A} \to \overline{AAB} \to \overline{AABA*}$, so the internal address is 1-3-5.

Conversely, the internal address 1-3-5 gives $\overline{A} \to \overline{AAB} \to \overline{AABA*}$.

Further applications of internal addresses and kneading sequences include:

A combinatorial proof that periodic parameter rays are landing in pairs at roots (Schleicher).

Homeomorphisms between sublimbs (Dudko–Schleicher).
1.5 Concrete and abstract combinatorial concepts

\[
\begin{align*}
\text{hyp. component, center, root} & \quad \rightarrow \quad \text{finite internal address} \\
\text{pair of conjugate per. angles} & \quad \rightarrow \quad *\text{-periodic kneading sequence} \\
\text{oriented Hubbard tree} & \quad \rightarrow \quad \text{abstract Hubbard tree}
\end{align*}
\]

The mapping is not injective: the same addresses are realized in sublimbs of equal denominators.

The mapping is not surjective: an internal address may be non-admissible.
1.6 Conversion algorithms:

admissible angled internal address

pair of conjugate periodic angles \rightarrow oriented Hubbard tree

Several algorithms for conversions between different combinatorial descriptions are given by Bruin and Schleicher.

The algorithms converting an internal address or kneading sequence to external angles are quite involved: growing of trees or constructing the Hubbard tree first.
2. The Stripping Algorithm:

For a given kneading sequence and (angled) internal address, the characteristic angles shall be computed.

This is done recursively for all periods in the internal address.

In the step from 1-\ldots-k to 1-\ldots-k-n, suppose that the center \( \hat{c} \) of period \( n \) belongs to the \( p/q \)-sublimb of the center \( c_0 \) of period \( k \). Then the angles of period \( qk \) are determined from those of period \( k \) by Douady tuning.

If \( n < qk \), \( \hat{c} \) will be behind the satellite component. Its angles are computed by iterating backwards.
2.1 Example of 1-3-5 or $AABA*$ with strips:

1-3 has the angles $\frac{1}{7}$ and $\frac{2}{7}$. Now $q = 2$, and 1-3-6 has angles $\frac{10}{63}$ and $\frac{17}{63}$.

Compare the dynamics of $c$ with period 6 to that of $\hat{c}$ with period 5:
- 5-periodic rays of $\hat{c}$ are not yet landing together for $c$.
- 6-periodic rays of $c$ are still landing together for $\hat{c}$.

Preimages of 6-periodic rays define stable strips in both planes. They are chosen according to the kneading sequence of $\hat{c}$. $U_1$ gives $\frac{5}{31}$ and $\frac{6}{31}$. 

![Diagram showing preimages of 6-periodic rays for $c$ and $\hat{c}$]
2.2 The Stripping Algorithm with strips:

For the step from 1-\ldots-k to 1-\ldots-k-n, start with the center $c$ and the angles $\theta_\pm$ of period $qk$.

0. The wake $W$ is bounded by the external rays for $\theta_\pm$.

1. Consider the strip $U_n = f_c^{-1}(W)$. Set $\mathbb{C} \setminus \overline{U_n} = A \cup B$.

2. For $j = n - 1$ down to $j = 1$, first check if $U_{j+1} \subset W$. Then the algorithm will be branching. $W$ is replaced with subsets of $W \setminus \overline{U_{j+1}}$.

3. For $j = n - 1$ down to $j = 1$, then choose a component of $f_c^{-1}(U_{j+1})$ for $U_j$ according to the kneading sequence of $1-\ldots-k-n$.

4. A branch is failing, if $U_1 \not\subset W$. Otherwise the unique $n$-periodic rays with angles $\Theta_\pm$ are recorded.
2.3 The Stripping Algorithm with angles:

For the step from 1-. . . -k to 1-. . . -k-n, start with the center c and the angles θ± of period qk.

0. The wake W corresponds to the interval (θ−, θ+).

1. In step n, compute the four angles Θ1 = θ−/2, Θ2 = θ+/2, Θ3 = (θ− + 1)/2, and Θ4 = (θ+ + 1)/2.

2. In steps j = n − 1 down to j = 1, first check if θ− < Θ1 < θ+. Then the algorithm is branching: (θ−, θ+) is replaced with (θ−, min{Θ1, Θ3}) in one branch and with (max{Θ2, Θ4}, θ+) in the other branch.

3. In steps j = n − 1 down to j = 1, then compute preimages. Case A: Θi is replaced with (Θi+1)/2 if 0 < Θi < θ− and with Θi/2 if θ+ < Θi < 1. It is vice versa in case B.

4. The branch is not failing if θ− < Θ1 < θ+. Then there are two unique n-periodic angles Θ± with Θ1 < Θ± < Θ2 and Θ3 < Θ± < Θ4.
2.4 Example of 1-3-5 or $\overline{AABA}$ with angles:

<table>
<thead>
<tr>
<th>$(\frac{10}{63}, \frac{17}{63})$</th>
<th>$W$</th>
<th>$[\frac{5}{31}, \frac{8}{31}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\frac{5}{63}, \frac{17}{126}) \cup (\frac{73}{126}, \frac{40}{63})$</td>
<td>$U_5$</td>
<td>$[\frac{3}{31}, \frac{4}{31}] \cup [\frac{18}{31}, \frac{19}{31}]$</td>
</tr>
<tr>
<td>$(\frac{34}{63}, \frac{143}{252}) \cup (\frac{73}{252}, \frac{20}{63})$</td>
<td>$U_4$</td>
<td>$[\frac{17}{31}, \frac{17}{31}] \cup [\frac{9}{31}, \frac{9}{31}]$</td>
</tr>
<tr>
<td>$(\frac{97}{126}, \frac{395}{504}) \cup (\frac{325}{504}, \frac{83}{126})$</td>
<td>$U_3$</td>
<td>$[\frac{24}{31}, \frac{24}{31}] \cup [\frac{20}{31}, \frac{20}{31}]$</td>
</tr>
<tr>
<td>$(\frac{97}{252}, \frac{395}{1008}) \cup (\frac{325}{1008}, \frac{83}{252})$</td>
<td>$U_2$</td>
<td>$[\frac{12}{31}, \frac{12}{31}] \cup [\frac{10}{31}, \frac{10}{31}]$</td>
</tr>
<tr>
<td>$(\frac{97}{504}, \frac{395}{2016}) \cup (\frac{325}{2016}, \frac{83}{504})$</td>
<td>$U_1$</td>
<td>$[\frac{6}{31}, \frac{6}{31}] \cup [\frac{5}{31}, \frac{5}{31}]$</td>
</tr>
</tbody>
</table>

The intervals of $U_1$ are approximately $(\frac{5.97}{31}, \frac{6.07}{31}) \cup (\frac{4.998}{31}, \frac{5.11}{31})$, determining unique angles $\Theta_+ = \frac{6}{31}$ and $\Theta_- = \frac{5}{31}$ of period 5.

In the third column, only 5-periodic angles are used, rounded to the interior of the intervals when necessary.
3. Early returns

The first example is given by 1-2-6-7-13-14 or $ABABAABABABABAB*$. Here $U_{10} \subset W$, so its preimages belong to $U_{14}$ instead of $A$ and $B$.

The Stripping Algorithm is branching to try both strips possible for $U_9$. The first branch is failing with the angles $\frac{6443}{16383}$ and $\frac{6444}{16383}$; the second branch is succeeding with the angles $\frac{6347}{16383}$ and $\frac{6348}{16383}$.

This phenomenon requires much consideration in the algorithm and the proof.
3.1 Oriented Hubbard trees; the correct branch with the angles $\frac{6347}{16383}$ and $\frac{6348}{16383}$, and the failing branch:
3.2 Oriented Hubbard trees; the failing branch and a similar Hubbard tree for a different center. It has the external angles $\frac{6443}{16383}$ and $\frac{6444}{16383}$. In the kneading sequence, there is an $A$ instead of a $B$ at the position of $c_9$:
4. The non-admissible example of 1-2-4-5-6 or $ABAAB^*$

This combinatorics is not admissible: there is no oriented Hubbard tree, external angle, or quadratic polynomial realizing these combinatorics.

Left: the Stripping Algorithm has an early return $U_4 \subset W$. Both branches are failing, since $U_1$ turns up on the wrong side of $W \setminus U_4$.

Right: the abstract Hubbard tree has a 3-periodic evil branch point.
5. The proof of the Stripping Algorithm shows:

1. Iterating the strips $U_j$ backwards is well-defined, since $c \notin U_{j+1}$. No strip is linked with the (possibly reduced) wake $W$.

2. If the algorithm does not branch, or if a branch does not fail, $U_1$ determines the desired characteristic angles of period $n$. Any other branch will be failing. The address $1-\ldots-k-n$ is realized.

3. If the algorithm is branching and all branches are failing, the address $1-\ldots-k-n$ will not be realized by a quadratic polynomial.
5.1 Pulling back is well-defined:

Show that no strip is containing $c$. Strips are disjoint or nested.

If there are several early returns, the earlier ones will not be before later ones; therefore they will not be linked with a reduced wake.
5.2 Non-failing branch gives correct angles:

In any case, comparing itineraries shows that $U_1 \subset W_0$ (original wake). So there are angles of period dividing $n$.

Forward iterates of rays are contained in the strips $U_j$. So the kneading sequence is the required one. In the case of an early return, note the components besides $f^{-1}_{c}(W) \subset U_n$ defining the kneading sequence. If the branch is not failing, the strips will be on the correct side.
5.3 Admissible address gives non-failing branch:

Show that the strips $U_j$, defined originally for $c$, do not bifurcate for parameters between $c$ and $\hat{c}$.

This is obvious when there is no early return, since a dynamic ray is bifurcating only when the critical value belongs to an image of this ray.

If there are early returns, there will be a unique branch point $y$ between $c$ and $z_1$, which corresponds to a Misiurewicz point $a$ between $c$ and $\hat{c}$. All possible image rays belong to other branches of $y$ or $a$, respectively.
6. Recapture of the critical point:

Recall the example of the center $\hat{c}$ of period 5 behind the center $c$ of period 6: in the latter plane, the preimages $z_j \in U_j$ represent the critical orbit of $f_{\hat{c}}$, except that $f_c(0) = c$ so $f_c(0) \neq z_1$.

Connecting $z_1, \ldots, z_5$ by a tree $T \subset K_c$ gives the oriented Hubbard tree of $f_{\hat{c}}$, but the mapping $f : T \rightarrow T$ is not the restriction of $f_c$: it must be modified in a neighborhood of 0.

In fact this modification turns $f_c$ into a topological polynomial $g$, which will be Thurston equivalent to $f_{\hat{c}}$. 
6.1 Recapture using Thurston Theorem

**Dfn:** Consider \( f_c(z) = z^2 + c \), and a preimage \( z_1 \neq c \) of 0, such that \( z_1, z_2 = f_c(z_1), \ldots, z_n = f_c^{n-1}(z_1) = 0 \) are distinct. Set \( \mathcal{P} = \{ z_1, z_2, \ldots, z_n, \infty \} \). Suppose \( \gamma \) is an arc from \( c \) to \( z_1 \), which is disjoint from \( \mathcal{P} \setminus \{ z_1 \} \). **Recapture** along \( \gamma \) gives a center \( \hat{c} \) of period \( n \) as follows:

1. Choose a disk neighborhood \( \Delta \) of \( \gamma \) disjoint from \( \mathcal{P} \setminus \{ z_1 \} \).
2. Choose a homeomorphism \( \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) with \( \phi(c) = z_1 \), which is the identity on \( \hat{\mathbb{C}} \setminus \Delta \).
3. The topological polynomial \( g = \phi \circ f_c \) is combinatorially equivalent to a unique \( f_{\hat{c}} \).

Construction is similar to Wittner **capture** and Timorin **regluing**. Based on

**Thm.:** (Thurston, Levy) A topological quadratic polynomial with periodic critical point is combinatorially equivalent to a unique polynomial.
6.2 Combinatorics and recapture

**Thm.:** Suppose $\hat{c}$ has the internal address $1\ldots-k-n$ and $c$ is the center of period $qk$ before it, bifurcating from $1\ldots-k$. If $q = 2$, $c$ may be the center of period $k$ as well. Recall the preimages of 0, $z_j \in U_j$ with $f_c^{n-j}(z_j) = 0$.

The arc $\gamma$ from $c$ to $z_1$ shall be homotopic to an arc disjoint from $U_j$ for $j = 2, \ldots, n$. Then $\hat{c}$ is obtained from $c$ by recapture along $\gamma$.

**Proof:** Choose $\Delta \subset W$ disjoint from $U_j$, $1 < j \leq n$, and $g = \phi \circ f_c$. Then $g = f_c : U_j \to U_{j+1}$ for $1 \leq j < n$. In $g^{-1}(U_1) \subset U_n$, choose an extended ray homotopic to its preimage. The quadratic polynomial equivalent to $g$ will have extended rays with the same cyclic order landing on the critical orbit.
6.3 Recapture with early return

Recall the example of 1-2-6-7-13-14 with the early return $U_{10} \subset W$. Work in the dynamic plane of $f_c$ with period 13 or 26.

If $\gamma$ goes from $c$ to $z_1$ directly, $g = \phi_0 \circ f_c$ will be equivalent to $f_{\hat{c}}$ for the center $\hat{c}$ with the correct internal address and the external angles $\frac{6347}{16383}$ and $\frac{6348}{16383}$.

If $\gamma$ turns once around $z_{10}$, $g = \phi_1 \circ f_c$ will be equivalent to $f_{c'}$, where the center $c'$ has the internal address 1-2-6-7-9-14 and the external angles $\frac{6443}{16383}$ and $\frac{6444}{16383}$.
6.4 More general curves:

Again, consider $\hat{c}$ with the address 1-2-6-7-13-14, $c'$ with 1-2-6-7-9-14, and $c$ with 1-2-6-7-13 or 1-2-6-7-13-26 before them.

Let $\gamma_m \subset W$ be an arc from $c$ to $z_1$ turning $m$ times around $z_{10}$. Then $g = \phi_m \circ f_c$ will be equivalent to $f_{\hat{c}}$ or to $f_{c'}$ depending on $m$.

Now $\phi_m \circ \phi_0^{-1}$ is isotopic to $D^{-m}$ for a Dehn twist $D$ about $z_1$ and $z_{10}$. Consider a Dehn twist $\hat{D}$ about $c_1$ and $c_{10}$. Then $\hat{D}^m \circ f_{\hat{c}}$ will be equivalent to $f_{\hat{c}}$ or to $f_{c'}$, respectively.

Compare this to the ...
6.5 Twisted Rabbit

Bartholdi–Nekrashevych have solved the Twisted Rabbit problem: $f_c$ is the Rabbit polynomial and $D$ is a Dehn twist about $c_1$ and $c_2$. Now consider $g = D^m \circ f_c$. Is it equivalent to the Airplane, the Rabbit, or its conjugate? All three types are realized for $m \in \mathbb{Z}$.

The images show that $g = D \circ f_c$ is equivalent to the Airplane, since the curves representing extended rays are homotopic to their preimages; the circular order determines the polynomial.