Convergence properties of the Thurston Algorithm for quadratic matings

Wolf Jung, Gesamtschule Brand, Aachen, Germany.

Mating is an operation to combine the dynamics of two complex quadratic polynomials, defining a rational map. In simple cases, it is obtained from the Thurston Algorithm by iterating a pullback map in Teichmüller space. When there are removable obstructions, it is customary to modify the algorithm. It will be shown that the usual algorithm does converge in a modified sense as well, which allows for a simpler implementation.

The images are made with Mandel, a program available from www.mndynamics.com .

1a. **Polynomial dynamics**

Iteration of $f_c(z) = z^2 + c$. Filled Julia set $\mathcal{K}_c = \{z \in \mathbb{C} \mid f_c^n(z) \not\to \infty\}$.

External rays $\mathcal{R}_c$ with rational angles land at (pre-)periodic points. The angle is doubled under iteration.
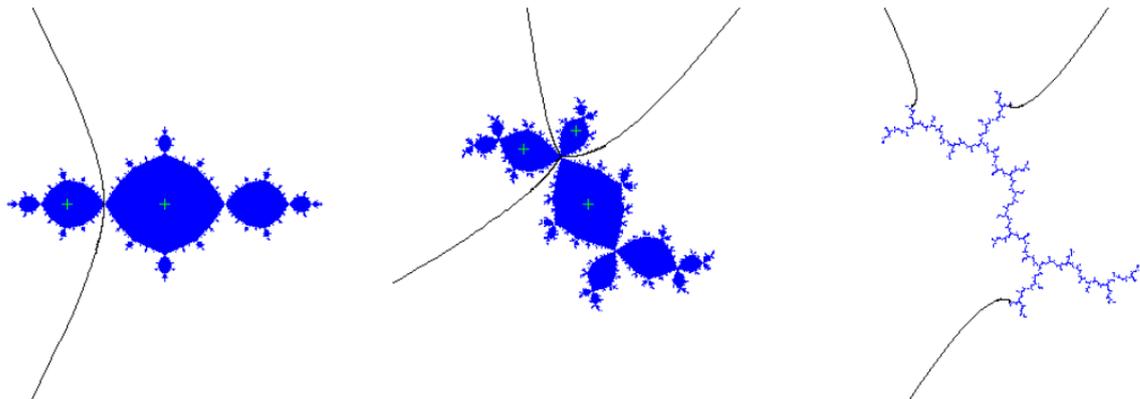

For the Basilica with $f(z) = z^2 - 1$, the relevant angles $1/3$ and $2/3$ are 2-periodic; the rays land at the same fixed point.

For the Rabbit the angles $1/7$, $2/7$, $4/7$ are 3-periodic.

For $f(z) = z^2 + i$ the critical value has the preperiodic angle $1/6$, which is mapped to $1/3$ and $2/3$.

1b. **Example of mating**

Rational maps or polynomials of higher degree have several critical points. The dynamics is more involved, and the parameter space is higher-dimensional. We may study one-dimensional slices and various combinations.
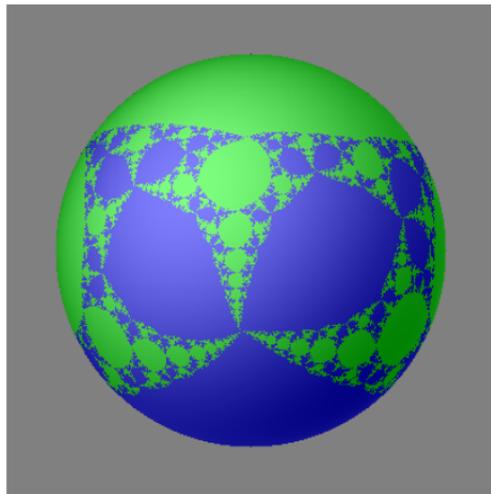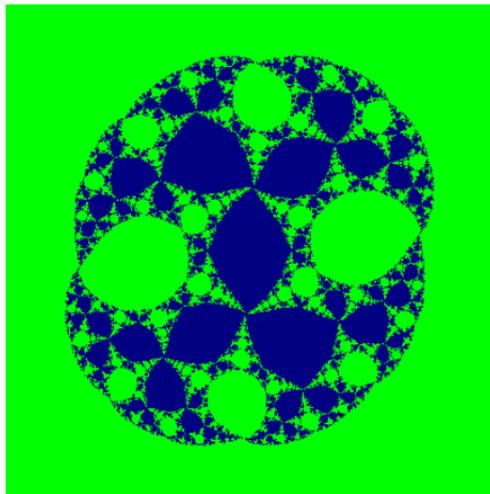
The first examples of mating were found by Douady–Hubbard. Here a Rabbit and a Basilica are glued along their boundaries, according to conjugate angles.

1c. **Definitions of mating**

Consider quadratic polynomials $P(z) = z^2 + p$ and $Q(z) = z^2 + q$ with locally connected Julia sets. The *topological mating* $P \coprod Q$ is defined by gluing the filled Julia sets according to conjugate angles, giving a branched covering of a topological sphere in good cases. If it is conjugate to a rational map f, this is the *geometric mating*.

The *formal mating* $g = P \sqcup Q$ is defined by mapping the two planes to half-spheres. In the postcritically finite case, an equivalent rational map f may be obtained from the Thurston Algorithm, at least for the essential mating $\widetilde{g}$. *The* **Rees-Shishikura-Tan Theorem** *ensures this to work, whenever the parameters p and q are not in conjugate limbs of the Mandelbrot set.*
The **Rees-Shishikura Theorem** *gives a semi-conjugacy from g to f, showing that the topological and geometric matings exist as well.*

Hyperbolic and parabolic maps are treated by surgery. Other postcritically infinite maps can be treated for certain families by using puzzle techniques.

2. The unobstructed example of Rabbit mates Basilica

2a. **Implementation of the Thurston Algorithm**

For the mating of $P(z) = z^2 + c_R$ and $Q(z) = z^2 - 1$, try to obtain the rational map f by pulling back approximate values of the postcritical points. The branch portrait $\infty \Rightarrow 1 \to \infty$, $0 \Rightarrow x_1 \to x_2 \to 0$ suggests

$$x_1' = \pm\sqrt{\frac{x_1 - x_2}{1 - x_2}} \quad x_2' = \pm\sqrt{x_1} \ ,$$

but how to determine the choice of signs? The Thurston Algorithm uses a pulllback of homeomorphisms ψ_n and a branched covering g. Under suitable conditions, the homeomorphisms converge in the Teichmüller space $\mathcal{T}$, so up to isotopy.

$$
\begin{array}{ccc}
S & \xrightarrow{\psi_0} & S_0 \\
g\uparrow & & f_0\uparrow \\
S & \xrightarrow{\psi_1} & S_1 \\
g\uparrow & & f_1\uparrow \\
S & \xrightarrow{\psi_2} & S_2 \\
g\uparrow & & f_2\uparrow \\
S & \xrightarrow{\psi_3} & S_3 \\
g\uparrow & & f_3\uparrow \\
& \xrightarrow{\ \cdots\ } &
\end{array}
$$

2b. Moduli space and Teichmüller space

topological sphere   Riemann sphere   hyperbolic metric
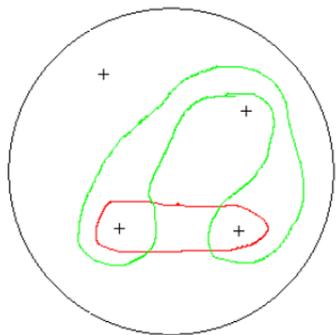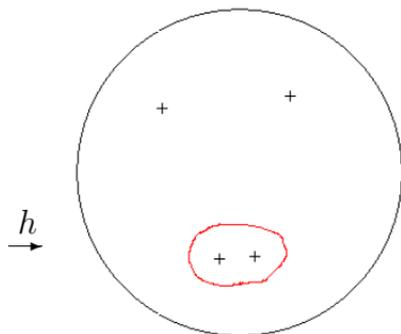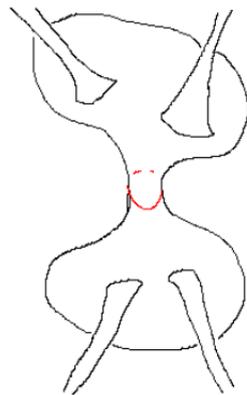
Teichmüller space contains different isotopy classes of homeomorphisms h, which may map to the same Riemann surface.

In the images, different curves are mapped to the same geodesic by different choices of $[h]$.

2c. Construction of the mating

The formal mating g has the dynamics of P on the lower half-sphere, and the dynamics of Q on the upper half-sphere. It is a postcritically finite branched covering.

*The **Thurston Theorem** says that, except when g is of type $(2, 2, 2, 2)$, an equivalent rational map f exists, if and only if g is unobstructed. f is the fixed point of the Thurston pullback map $\sigma_g : \mathcal{T} \to \mathcal{T}$; it is globally attracting.*

Here Teichmüller space is used both to define a unique pullback, and to obtain convergence results.

In our example, the **Rees-Shishikura-Tan Theorem** shows that g is unobstructed, so the rational map f exists and $f_n \to f$.

2d. **Slow mating**

Instead of pulling back triangulations or medusas, *slow mating* employs a path in moduli space. The images illustrate this algorithm for the same example, $P(z) = z^2 + c_R$ and $Q(z) = z^2 - 1$. Here the polynomial Julia sets are pulled back together with the marked points.
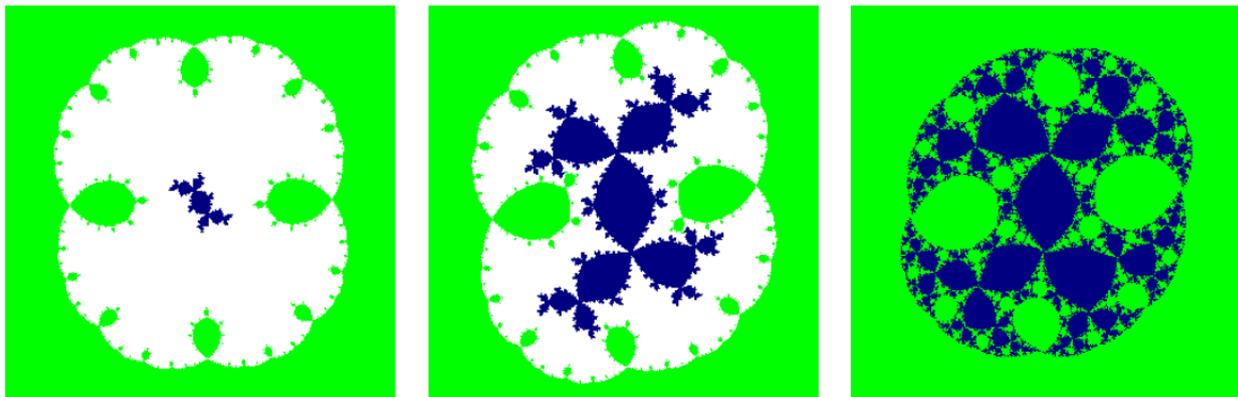
Using this algorithm, movies of mating have been made by Xavier Buff and Arnaud Chéritat. Note that the algorithm itself is quite fast, but the movie slows it down to illustrate the process.

3. The obstructed example of $z^2 + \mathrm{i}$ mates Basilica

3a. **Slow mating**

The images illustrate the algorithm of *slow mating* for $P(z) = z^2 + \mathrm{i}$ and $Q(z) = z^2 - 1$. Again, the polynomial Julia sets are pulled back together with the marked points.
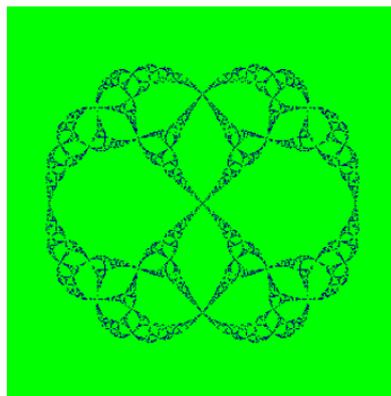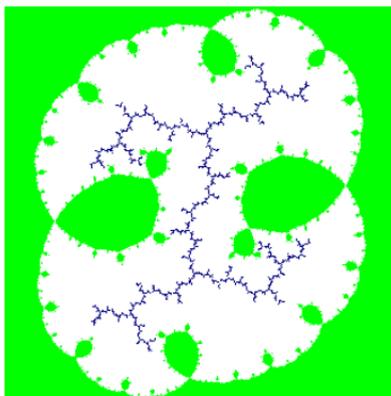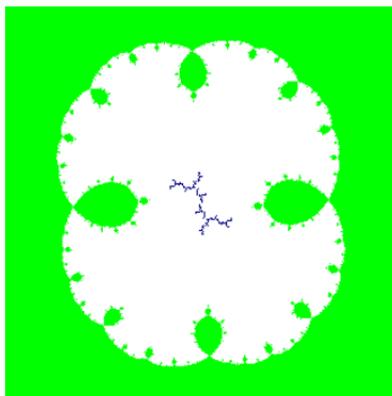
The 2-periodic points of P are identified with each other and with a fixed point of Q in the limit.

3b. **Implementation of the Thurston Algorithm**

For the mating of $P(z) = z^2 + i$ and $Q(z) = z^2 - 1$, try to obtain the rational map f by pulling back approximate values of the postcritical points. The branch portrait $\infty \Rightarrow 1 \to \infty,\ 0 \Rightarrow x_1 \to x_2 \leftrightarrow -x_1$ suggests

$$x_1' = \pm\sqrt{\frac{x_1 - x_2}{1 - x_2}} \quad x_2' = \pm\sqrt{\frac{2x_1}{1 + x_1}}\ .$$

Again, the choice of signs is determined from pulling back a path in moduli space, or isotopy classes of homeomorphisms in Teichmüller space. The isotopy class of g provides the necessary topological-combinatorial information.

$$
\begin{array}{ccc}
S & \xrightarrow{\ \psi_0\ } & S_0 \\
g\big\uparrow & & \big\uparrow f_0 \\
S & \xrightarrow{\ \psi_1\ } & S_1 \\
g\big\uparrow & & \big\uparrow f_1 \\
S & \xrightarrow{\ \psi_2\ } & S_2 \\
g\big\uparrow & & \big\uparrow f_2 \\
S & \xrightarrow{\ \psi_3\ } & S_3 \\
g\big\uparrow & & \big\uparrow f_3 \\
& \xrightarrow{\ \cdots\ } &
\end{array}
$$

3c. **Removable obstruction**

Recall the **Thurston Theorem***: except when g is of type $(2, 2, 2, 2)$, an equivalent rational map f exists, if and only if g is unobstructed. f is the fixed point of the Thurston pullback map $\sigma_g : \mathcal{T} \to \mathcal{T}$; it is globally attracting.*

An obstructing multicurve implies that certain marked points will be identified under the pullback, which means divergence in Teichmüller space. In our example, the 2-periodic points of P are identified with a fixed point of Q. The corresponding obstruction is understood as surrounding the rays with angles $1/3$ and $2/3$.

So for the formal mating g of $z^2 + \mathrm{i}$ and $z^2 - 1$, the Thurston Algorithm diverges, and there is no equivalent rational map f.

3d. **The essential mating**

Suppose P and Q are postcritically finite, not from conjugate limbs, and consider the formal mating g. According to [Tan92], obstructions contain removable Lévy-cycles and correspond to ray-equivalence classes with more than one postcritical point. Pinching these defines the essential mating $\widetilde{g}$, which is equivalent to a quadratic rational map f.

So by applying the Thurston Theorem to the essential mating $\widetilde{g}$ instead of the formal mating g, we can define f. But there are two drawbacks:

• **Ray-equivalence classes may be quite involved, and pinching them algorithmically shall be impractical.**

• **The movies of slow mating or equipotential gluing are based on the unmodified formal mating.**

3e. **Alternative convergence result**

Theorem: *Although the pullback for the formal mating diverges in Teichmüller space, we have convergence $f_n \to f$ and two marked points are identified in the limit, if and only if they belong to the same ray-equivalence class.*

Moreover, we may mark non-postcritical points in addition, to show that *all (pre-)periodic points of P and Q converge.* Here additional obstructions are created and used as a tool to prove convergence.

The proof requires a combination of global and local techniques, to show that the points get together and have a limit.

4. Proof of the convergence statement

4a. **Noded Riemann surfaces**

topological sphere   Riemann sphere   hyperbolic metric
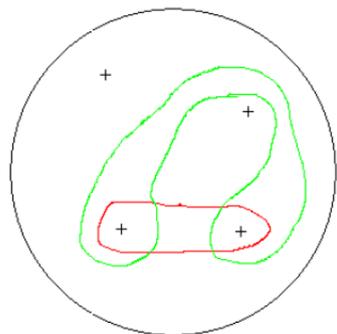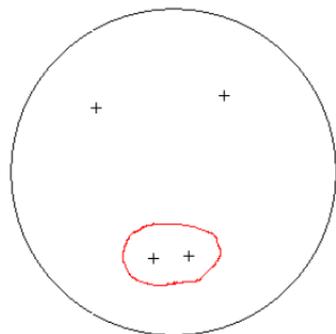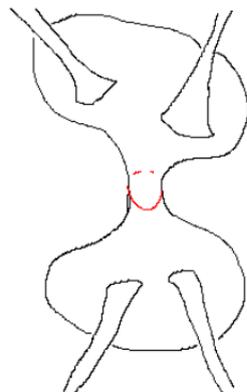
Teichmüller space contains different isotopy classes of homeomorphisms h, which may map to the same Riemann surface.

short hyperbolic geodesic $\Leftrightarrow$ large modulus $\Leftrightarrow$ marked points get close

Compactification of moduli space by adding noded surfaces: smooth components intersecting transversely, or collection of spheres with additional marked points, since the hyperbolic metric shall be singular at the nodes.

4b. Augmented Teichmüller space

$\mathcal{T}$ is complete with respect to Teichmüller distance d_T, not w.r.t Weil-Petersson distance d_{WP}. Augmented Teichmüller space is the completion; it is not locally compact. There are continuous projections $\pi : \mathcal{T} \to \mathcal{M}$ and $\pi : \widehat{\mathcal{T}} \to \widehat{\mathcal{M}}$. Only the former is an unbranched covering.

We have stratified spaces $\widehat{\mathcal{T}} = \bigcup \mathcal{S}_\Gamma$ and $\widehat{\mathcal{M}} = \bigcup \mathcal{S}_{[\Gamma]}$. Properties of $\widehat{\mathcal{T}}$:

- $\mathcal{S}_\Gamma$ is approximated only from $\mathcal{S}_{\widetilde{\Gamma}}$ with $\widetilde{\Gamma} \subset \Gamma$.

- Using a normalization, coordinates of marked points are continuous on $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{T}}$.

- The length of geodesics is continuous on $\widehat{\mathcal{T}}$; the collection is injective.

4c. **Selinger's results**

Extension Theorem: *The Thurston pullback map σ_g extends to a Lipschitz-continuous map on $\widehat{\mathcal{T}}$, which has an explicit description on noded spheres.*

Canonical obstruction Theorem: *There is a unique canonical obstruction Γ_g, such that its curves are pinched under the iteration $\sigma_g^n(\tau_0)$ for $\tau_0 \in \mathcal{T}$.* [Pilgrim, hyperbolic orbifold]

On $\mathcal{T}$, $\sigma_g^n(\tau_0)$ diverges if $\Gamma_g \neq \emptyset$.

On $\widehat{\mathcal{T}}$, $\sigma_g^n(\tau_0)$ accumulates at most on $\mathcal{S}_{\Gamma_g} \subset \widehat{\mathcal{T}}$ and $\pi(\sigma_g^n(\tau_0))$ accumulates on a compact subset of $\mathcal{S}_{[\Gamma_g]} \subset \widehat{\mathcal{M}}$, which depends on $\tau_0 \in \mathcal{T}$.

Characterization Theorem: *Γ_g is characterized as the smallest simple obstruction, such that first-return maps of component spheres are homeomorphisms or unobstructed (or special maps of type $(2, 2, 2, 2)$).*

4d. **Proof of the convergence statement (1)**

In our example, a loop γ around the ray-equivalence class with angles $1/3$ and $2/3$ forms a simple obstruction Γ.

By the Selinger characterization, Γ is the canonical obstruction Γ_g, thus pinching. It remains to show that the clusters of points do not wander.

Step 1: global result in augmented Teichmüller space, Selinger
By the Canonical Obstruction Theorem, $\tau_n = \sigma_g^n(\tau_0)$ has a subsequence with $d_{WP}\left(\sigma_g^i(\tau_{n_k}), \sigma_g^i(\widehat{\tau}_t)\right) < \varepsilon(t)$ for $0 \leq i \leq t$ with suitable $\widehat{\tau}_t \in \mathcal{S}_{\Gamma_g} \subset \widehat{\mathcal{T}}$, such that $\pi(\widehat{\tau}_t)$ is an accumulation point of $\pi(\tau_n)$. This shadowing property is obtained for a finite time only, since σ_g need not be contracting with respect to d_{WP}.

In our example, the small sphere contributes dimension 0 to the stratum. An interplay between d_T and d_{PW} yields uniform estimates, showing that $\sigma_g^i(\widehat{\tau}_t)$ and $\sigma_g^i(\tau_{n_k})$ are close to the fixed point of $\sigma_{\widetilde{g}}$ on the central component for suitable i and t.

4e. **Proof of the convergence statement (2)**

Step 2: local result in coordinate space

The limit does not carry over immediately to $\sigma_g^i(\tau_{n_k})$, because $i \leq t$. We shall see below that there is local attraction to the desired points, so the proof is completed by choosing i and t large enough and considering a suitable short path between $\sigma_g^{i-1}(\tau_{n_k})$ and $\sigma_g^i(\tau_{n_k})$.

In our example, we want to show that $x_1 \to -2$, while both $x_2 \to 2$ and $x_3 = -x_1 \to 2$. Consider the derivative in the coordinates x_1, x_2 and the transformation $u_1 = x_1$, $u_2 = x_2 - (-x_1)$, respectively:

$$\begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1/2 & 3/4 \\ 0 & 3/4 \end{pmatrix}$$

The eigenvalue $-1/2$ comes from $\sigma_{\widetilde{g}}$, while $3/4$ is related to $f'(2) = -4/3$. Since both have modulus < 1, there is an attraction for the path in a neighborhood of $x_1 = -2$, $x_2 = 2$.

4f. **Proof of the convergence statement (3)**

More generally, consider P, Q postcritically finite, not from conjugate limbs, and the postcritical set $\mathcal{P}$ of the formal mating g. Exclude type $(2, 2, 2, 2)$.

a) If no two points of $\mathcal{P}$ are ray-equivalent, we have convergence of the unmodified Thurston Algorihm σ_g on $\mathcal{T}$ and $\mathcal{M}$.

b) If every ray-equivalence class contains at most 2 points of $\mathcal{P}$, we have convergence on $\widehat{\mathcal{T}}$ and $\widehat{\mathcal{M}}$.

c) If some ray-equivalence class contains at least 3 points of $\mathcal{P}$, we still have convergence of marked points.

In all cases, the rational maps converge $f_n \to f$.

So the example discussed above was of case b). The **proof** in case c) is based on the product structure of the canonical stratum: use the arguments above on the central sphere. The pullback on other spheres is non-trivial, but in the chosen normalization these correspond to small disks.

5. Conclusion

The general convergence theorem implies that slow mating and equipotential gluing do converge, $f_n \to f$, even when there are removable obstruction, which prevent convergence in Teichmüller space. Thus slow mating can be implemented without encoding the topology of postcritical ray-equivalence classes.

5a. **Related results**

- Implementation and convergence of anti-mating, captures, precaptures, twists, spider algorithm ...

- There are 30 matings of nine kinds, from non-conjugate limbs, such that $\widetilde{g}$ is of type $(2, 2, 2, 2)$. Then a rational f equivalent to $\widetilde{g}$ exists.

- Non-convergence of slow mating in that case (joint work with Arnaud Chéritat).

Thank you.

5b. **Unrelated results**

- Elementary examples of multiply shared matings and of mating discontinuity.

- For matings of conjugate polynomials, the leading eigenvalue of the Thurston matrix is related to core entropy.

- Explicit treatment of a twisted $(2, 2, 2, 2)$-map.

- Explicit construction of Hurwitz equivalences.

- Constructing topological matings by bounding ray connections.

- Hausdorff obstructions related to renormalization.

General convergence statement

Theorem: *Suppose g is a bicritical Thurston map of degree d, and Γ is a completely invariant multicurve such that:*

- *All components of $S \setminus \Gamma$ except C are disks; the periodic disks are mapped homeomorphically.*

- *C is mapped to itself with degree d; the essential map $\widetilde{g}$ is equivalent to a rational map f.*

Using a normalization at 0, 1, ∞, such that no disk contains two of them, the Thurston Algorithm for the unmodified map g satisfies:

- *The curves of Γ are pinched, so $[\psi_n]$ diverges in $\mathcal{T}$.*

- *If f is not of type $(2, 2, 2, 2)$, we have $f_n \to f$. The marked points converge to (pre-)periodic points of f, identified according to the disks.*

5d. **Convergence of Julia sets**

Consider the pullback of Julia sets in this situation:

Theorem: *The pullbacks $\mathcal{J}_n$ of $\partial\mathcal{K}_p$ and $\partial\mathcal{K}_q$ converge to the Julia set $\mathcal{J}$ of f with respect to Hausdorff distance.*

To show $\mathcal{J}$ is an an ε-neighborhood of $\mathcal{J}_n$, take a finite set of repelling periodic points and apply the convergence statement.

To see that $\mathcal{J}_n$ is in an ε-neighborhood of $\mathcal{J}$, note that f has only finitely many Fatou components of diameter $\geq \varepsilon$, and the Boettcher conjuagtions converge.

In the case of equipotential gluing, the holomorphic motion $\psi_t \circ \psi_0^{-1}$ is expected to converge to a semi-conjugation. *If both P and Q are hyperbolic, this can be shown* by noting that f_n is expanding with respect to the orbifold metric.
— What about non-hyperbolic cases?
— What about postcritically infinite cases?