Lattès maps and quadratic matings

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Dedicated to Mitsuhiro Shishikura for his $2^1 \cdot (2^5 - 1)$nd birthday

Abstract

In complex dynamics, mating is an operation to construct or to describe a postcritically finite rational map by combining two polynomials. When P and Q are from non-conjugate limbs of the Mandelbrot set $\mathcal{M}$, the formal mating $g = P \sqcup Q$ has only removable obstructions according to Tan Lei, and the essential mating $\widetilde{g}$ is unobstructed. The Thurston Theorem implies that there is an equivalent rational map f, unless $\widetilde{g}$ is of Lattès type (2, 2, 2, 2): then we need to construct an affine lift and check its eigenvalues.

In the present paper, the combinatorics of ray-equivalence classes is used to show that there are nine kinds of matings with $\widetilde{g}$ of type (2, 2, 2, 2). The Shishikura Algorithm yields the affine lifts, and there is an equivalent rational map f in each case. So the non-conjugate-limbs condition is sufficient for all postcritically finite quadratic matings.

1 Introduction

In complex dynamics, a classical topic is the iteration of quadratic rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, especially postcritically finite maps: the orbits of the critical points $z = 0$ and $z = \infty$ are periodic or preperiodic. The iteration f^n is stable on the Fatou set $\mathcal{F}_f$ and chaotic on the Julia set $\mathcal{J}_f = \widehat{\mathbb{C}} \setminus \mathcal{F}_f$. For a polynomial $P(z) = z^2 + p$, the fixed point $z = \infty$ is superattracting, and the filled Julia set $\mathcal{K}_p = \{z \mid P^n(z) \not\to \infty\}$ satisfies $\mathcal{J}_p = \partial\mathcal{K}_p$. The Mandelbrot set $\mathcal{M}$ is the set of parameters p, such that $\mathcal{K}_p$ is connected. For a postcritically finite parameter p, $\mathcal{K}_p$ is locally connected and the Carathéodory loop $\gamma_p : S^1 \to \mathcal{J}_p$ is continuous and a semi-conjugation with $P(\gamma_p(t)) = \gamma_p(2t)$ [19]. Geometrically, $\gamma_p(t)$ is the landing point of an external ray (or dynamic ray) $\mathcal{R}_p(t)$ with angle $2\pi t$ at ∞. See Figure 1 left.

Some rational maps f may be described by mating two polynomials P and Q: the filled Julia sets $\mathcal{K}_p$ and $\mathcal{K}_q$ are glued together along their boundaries, such that $\gamma_p(t)$ is identified with $\gamma_q(-t)$, which defines a topological sphere under appropriate conditions. If the resulting topological map is conjugate to f, this gives a semi-conjugation $\gamma_f : S^1 \to \mathcal{J}_f$ from angle-doubling. When both P and Q are critically preperiodic, the filled Julia sets have empty interiors and we cannot recognize the mating from an image of $\mathcal{J}_f = \widehat{\mathbb{C}}$; the Peano curve $\gamma_f : S^1 \to \widehat{\mathbb{C}}$ may be visualized by approximations or finite subdivision rules [18, 2]. — Here are some basic definitions and constructions, see Section 2 or [8, 11] for more explanations:

- A topological space $\mathcal{K}_p \coprod \mathcal{K}_q$ is defined by identifying $\gamma_p(t)$ with $\gamma_q(-t)$. If it is Hausdorff and homeomorphic to the sphere S^2, the *topological mating* $P \coprod Q : S^2 \to S^2$ is defined up to topological conjugacy [22].

- If there is a conjugate rational map $f \cong P \coprod Q$, this is the *geometric mating*.

While these definitions are fairly easy to state, it is hard to show directly, that $P \coprod Q$ or f exists [8]. In the postcritically finite case, this is done as follows:

- Define the *formal mating* $g = P \sqcup Q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that it is topologically conjugate to $P : \mathbb{C} \to \mathbb{C}$ on the lower half-sphere and to $Q : \mathbb{C} \to \mathbb{C}$ on the upper half-sphere, and such that the images of the rays $\mathcal{R}_p(t)$ and $\mathcal{R}_q(-t)$ meet at $\exp(\mathrm{i}2\pi t)$. See Figure 1 top right.

- There may be ray-connections between postcritical points; collapsing these gives the *essential mating* $\widetilde{g} : S^2 \to S^2$, which is a branched cover with fewer postcritical points, a Thurston map. See Figure 1 bottom right. Under the assumption of Theorem 1.1 below, $\widetilde{g}$ does not have a multicurve obstruction (or Thurston obstruction) according to Tan Lei [33, 34].

- Now the *combinatorial mating* f is a rational map combinatorially equivalent (or Thurston equivalent) to the essential mating, $f \sim \widetilde{g}$, which means f is topologically conjugate to $\widetilde{g}$ up to isotopy relative to the postcritical set.

- Since f is expanding in a neighborhood of $\mathcal{J}_f$, a semi-conjugation from g to f is obtained, which collapses each ray-equivalence class to a point according to Mary Rees and Mitsuhiro Shishikura [32]. So the quotient $\mathcal{K}_p \coprod \mathcal{K}_q$ is a sphere and the topological mating $P \coprod Q$ is conjugate to f, which is a geometric mating at last.

Each postcritically finite parameter $p \neq 0$ belongs to a limb $\mathcal{M}_{k/r}$ of the Mandelbrot set $\mathcal{M}$, such that the fixed point $\alpha_p \in \mathcal{K}_p$ has the rotation number k/r. If q belongs to the conjugate limb $\mathcal{M}_{-k/r}$, a mating $\cong P \coprod Q$ cannot exist, because there are several ray connections between α_p and α_q, so the quotient space $\mathcal{K}_p \coprod \mathcal{K}_q$ is not a sphere. Conversely, we have:

Theorem 1.1 (Rees–Shishikura–Tan)
Suppose $p, q \in \mathcal{M}$ are postcritically finite and not in conjugate limbs. Then the combinatorial mating $f \sim \widetilde{g}$ exists, and it is a geometric mating $f \cong P \coprod Q$.

The third step above, from the essential mating $\widetilde{g}$ to the combinatorial mating f, is based on the fundamental theorem of Bill Thurston [6, 7, 4], which comes in two flavors. The exceptional case concerns maps $\widetilde{g}$ of orbifold type (2, 2, 2, 2), which means there are four postcritical points, and no critical point is postcritical:

- Suppose a Thurston map $\widetilde{g}$ is not of type (2, 2, 2, 2). Then there is a contracting map $\sigma_{\widetilde{g}}$ on a Teichmüller space, whose fixed point defines a rational map $f \sim \widetilde{g}$. If there is no multicurve obstruction, the iteration $\sigma_{\widetilde{g}}^n(\tau_0)$ is uniformly contracting, so the fixed point exists.

- When $\widetilde{g}$ is of type (2, 2, 2, 2), $\sigma_{\widetilde{g}}$ is not contracting. Now $\widetilde{g}$ is covered by a real affine map on the torus $\mathbb{R}^2/\mathbb{Z}^2$; if this map is similar to a complex affine map, then $\widetilde{g}$ is combinatorially equivalent to a rational map f, a Lattès map.

There are formal matings $g = P \sqcup Q$, such that the essential mating $\widetilde{g}$ is of type $(2,\,2,\,2,\,2)$; then absence of multicurve obstructions is not enough to construct an equivalent rational map. This exceptional case was not treated completely in [33, 34]. It seems Tan Lei believed at the time, that there are just two examples to consider. I wrote to her in 2016 to ask about this, only to learn that she had passed away. — John Milnor [18] has given seven examples altogether, which he attributed to Mitsuhiro Shishikura in part. Each of these gives a rational map in fact, but the possibility of further examples constitutes a gap in the proof of Theorem 1.1.

	$L(w) = \eta w + \kappa$	$\rho =$	$f_c\,,\ F_u$	mating
a)	$\kappa = 0,\ \eta^2 = -2$	-1	$u = 1-\sqrt{2}$	$f \simeq 1/12 \coprod 5/12$
b)	$\kappa = 0,\ \eta^2 = -2\mathrm{i}$	i	$c = \mathrm{i}$	$f \cong 1/4 \coprod 1/4$
				$f \simeq 23/28 \coprod 13/28$
				$f \simeq 53/60 \coprod 29/60$
c)	$\kappa = 0,\ \eta^2 = \frac{-3+\sqrt{7}\mathrm{i}}{2}$	$\frac{-3-\sqrt{7}\mathrm{i}}{4}$	$c = \frac{1+\sqrt{7}\mathrm{i}}{2}$	$f \cong 3/14 \coprod 3/14$
				$f \simeq 3/14 \coprod 1/2$
				$f \simeq 5/6 \coprod 1/2$
				$f \cong 1/6 \coprod 5/14$
d)	$\kappa = 1/2,\ \eta^2 = \frac{-3-\sqrt{7}\mathrm{i}}{2}$	$\frac{-3+\sqrt{7}\mathrm{i}}{4}$	$c = \frac{1+\sqrt{7}\mathrm{i}}{4}$	$f \cong 1/6 \coprod 1/6$

Table 1: Nine examples of matings $f \cong P \coprod Q$ are given by external angles of p and q. See Sections 3.4 and 3.5 for the notation of rational maps f_c and F_u. Here $\cong$ means the rational map f conjugate to the topological mating has a standard normalization, $\gamma_f(0) = 1$, and $\simeq$ indicates that a rotation of the fixed points is applied in addition.

According to Table 1 b), there are four more representations of $f_{\mathrm{i}} \cong 1/4 \coprod 1/4$, of two kinds, which answers a question in [18]. Moreover, we shall see that these nine examples are the only ones (up to obvious transformations) of type $(2,\,2,\,2,\,2)$, and check that each of these produces a rational map according to the exceptional case of the Thurston Theorem. — This completes the proof of Theorem 1.1:

Theorem 1.2 (Lattès matings, following Shishikura)
1. *There are thirty formal matings $g = P \sqcup Q$ of quadratic polynomials, such that the essential mating $\widetilde{g}$ has orbifold type $(2,\,2,\,2,\,2)$, and the parameters p and q are not in conjugate limbs of the Mandelbrot set $\mathcal{M}$. Up to complex conjugation and interchanging P and Q, these matings are represented by the nine matings given in the table above.*

2. *In each case, the essential mating $\widetilde{g}$ is combinatorially equivalent to a rational map f, which is given in the table as well. So f is a geometric mating in fact, conjugate to the topological mating $P \coprod Q$.*

Both items are proved combinatorially. For item 2, a key step is the construction of a curve through the postcritical points of $\widetilde{g}$, such that its preimage is understood up to homotopy; then the affine lift is obtained explicitly and its eigenvalues are found to be not real. In Section 2, this is explained in detail for the example $1/12 \coprod 5/12$. A self-contained discussion of rational maps and Thurston maps of Lattès type $(2,\,2,\,2,\,2)$ is given in Section 3. Item 1 is proved in Section 4: although a ray-equivalence class may contain indirect ray connections, any fixed ray-equivalence class must contain a fixed point of P or Q, and there are restrictions on periods and orbits within a limb, which exclude higher rotation numbers. In Section 5, affine lifts are constructed from the Shishikura algorithm for the remaining examples, proving item 2. Related results are referenced in Section 6; for the Lattès map of type (2, 4, 4), it is shown that there are precisely six matings of three kinds.

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2 A worked out example of mating

The various definitions of rays, ray connections, and matings are explained for the example $1/12 \coprod 5/12$. A curve through the postcritical points and its preimage curve are constructed, and used to lift the essential mating to an affine map, from which an equivalent rational map is constructed. These techniques will be applied to other examples in Section 5. The general discussion of Lattès maps in Section 3 refers to this explanation as well.

2.1 The formal mating

landing in general and here

alpha limbs beta endpoints

formal, rays and ray connections

topological and essential (briefly)

2.2 The essential mating

formal and essential are tmaps

Figure above, right top to bottom by identifications only of ray connections not classes, additional half rays for pseudo, or rather moving equator

because topological and obstructions

[28] for canonical, [11] for equivalence

second step identify preimages, no invariant curve or set, additional crossings and a loop

note: collaps for tilde g, move pseudo eq in addition

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\chi_0} & S^2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \widehat{g}} \\ \widehat{\mathbb{C}} & \xrightarrow[\chi_0]{} & S^2 \end{array} \qquad \begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{\chi_1} & S^2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \widetilde{g}} \\ \widehat{\mathbb{C}} & \xrightarrow[\chi_0]{} & S^2 \end{array} \tag{1}$$

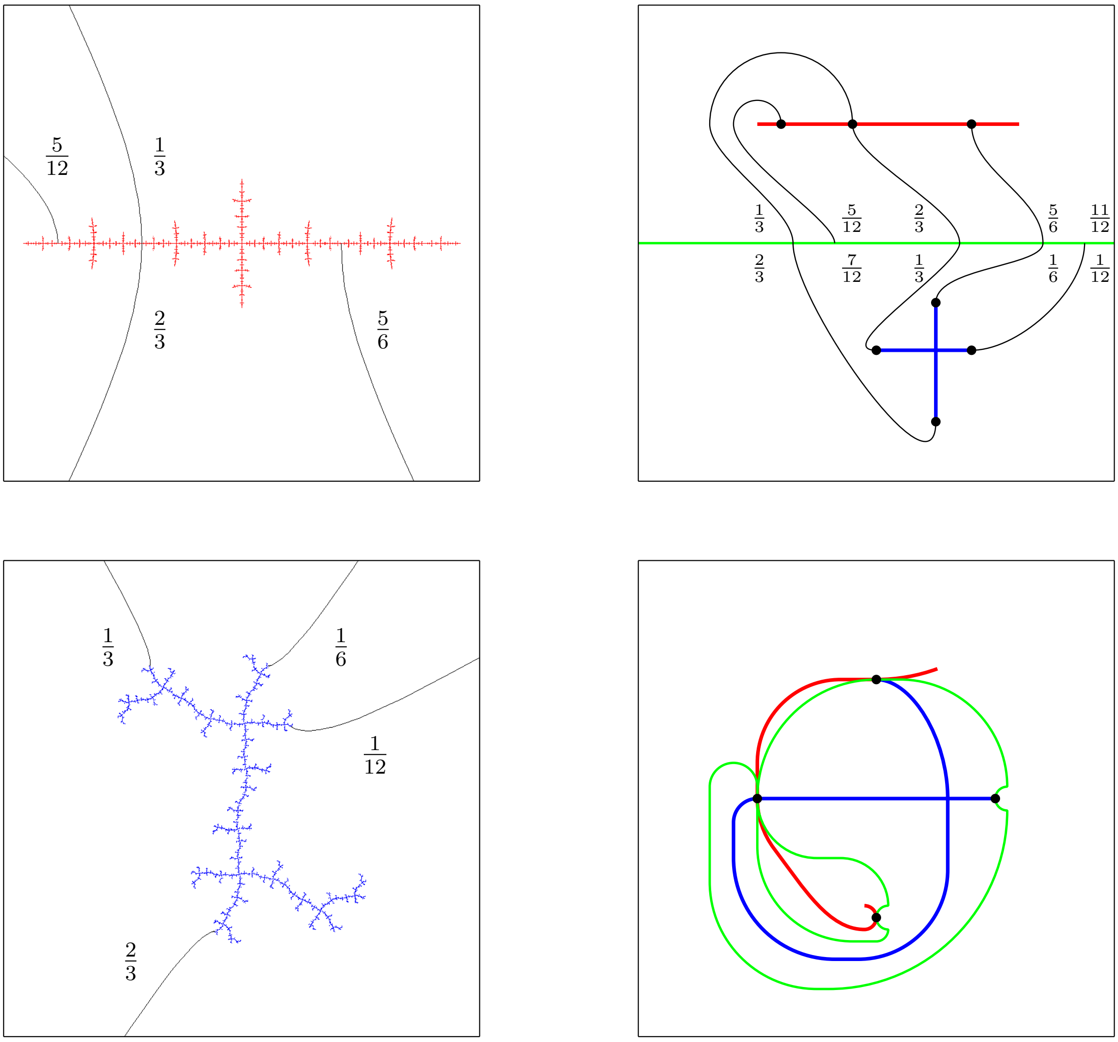

Figure 1: left julia sets, right equator seen from inside at -i, and pseudo-equator.

Remark 2.1 (Comparison to the notion of a pseudoequator)
pseudoeq Meyer [15, 16], for fn expanding and for unmating, simply con when direct raycons, multicon: Kameyama, Meyer, Wilkerson [14, 36]

here not pseudoiso required, just simply and understand preimage

2.3 A simple curve through the postcritical points

now lulia weg and colors

modification an einer stelle

dadurch simple und preimage nur zwei crossings kein loop

beachte alpha 1/2, kurven vertauscht

(figure as high up as possible)

(text from above as much as possible here)

Remark 2.2 (Laminations and the Shishikura algorithm)
In Figure 3 simplified and as round as possible. For lift convenient, also easier to construct in general: do not deform equator but look identifications from laminations. Just mark choice of angles on the circle, draw ray connections for required

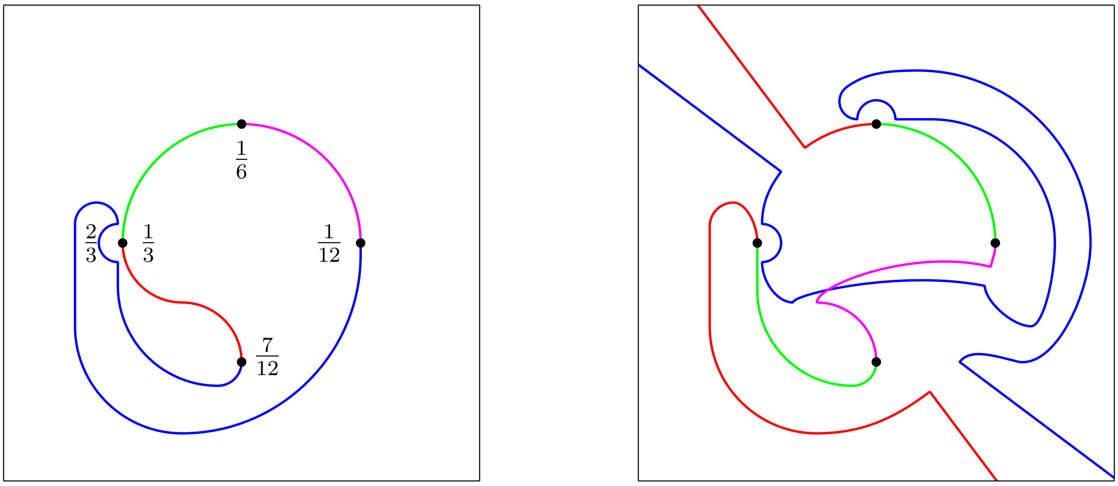

Figure 2: modified from pseudo curve, crossing at infty.

preimages or images, identify. related to alternative definition.

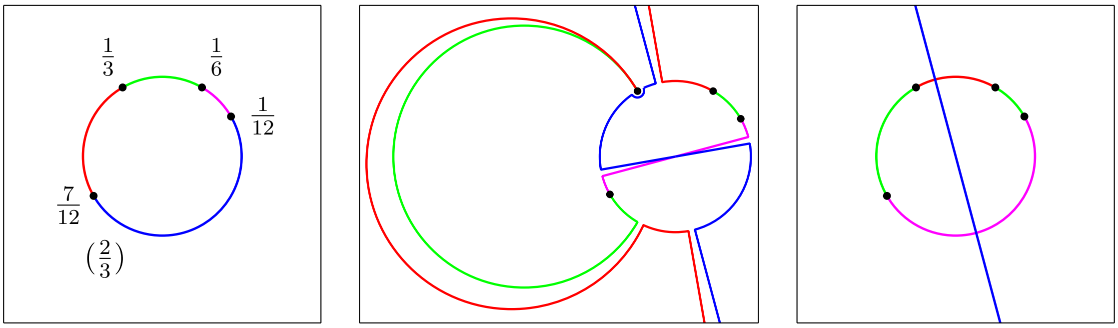

Figure 3: more circular and nebenbei laminated, and simplified, note which crpts on edges, convention, no longer 0 infty, even not meaningful, note homo rel P. — keep in mind two ideas, use of angles and orthocircles, and dealing with pinching points

2.4 Lifting to an affine map

lattice and sublattice and minus, pillowcase, weierstrass holo convenient

normalization circle

point is: branched cover, top map unbranched, ref orbifold

here explicitly by reflections, no general result used

easy from simplified image, in general additional homotopy from curves to lines, note precomposition map cf curves

matrix explicitly, det and eigenvalues

lift also in [SY] and Meyer peano p. 8

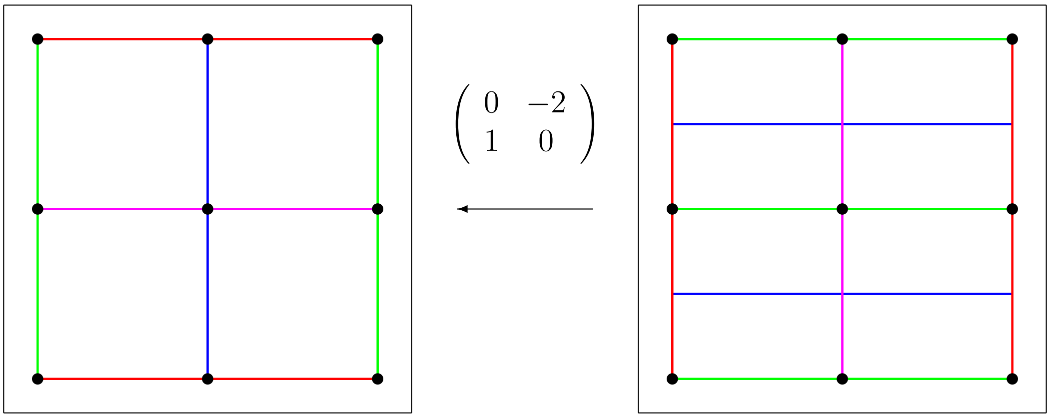

Figure 4: straightened by lift, preimage after homotopy, then equivalent to affine.

$$\begin{array}{ccccccccc}
S^2 & \longrightarrow & \widehat{\mathbb{C}} & \longleftarrow & \mathbb{C} & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathbb{C}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^2 & \longrightarrow & \widehat{\mathbb{C}} & \longleftarrow & \mathbb{C} & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{\mathbb{C}}
\end{array} \tag{2}$$

2.5 The combinatorial mating

eigenvalues, theorem not real ok or fp mobius, here explicitly complex affine
unique pcf quadratic
commuting diagram, solving beltrami

$$\begin{array}{ccccccc}
\mathbb{C} & \longleftarrow & \mathbb{C} & \longrightarrow & \widehat{\mathbb{C}} & \longrightarrow & \widehat{\mathbb{C}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C} & \longleftarrow & \mathbb{C} & \longrightarrow & \widehat{\mathbb{C}} & \longrightarrow & \widehat{\mathbb{C}}
\end{array}
\qquad
\begin{array}{ccc}
S^2 & \xrightarrow{\varphi_1} & \widehat{\mathbb{C}} \\
{\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle f} \\
S^2 & \xrightarrow[\varphi_0]{} & \widehat{\mathbb{C}}
\end{array} \tag{3}$$

2.6 The geometric mating

Remark 2.3 ()
beltrami not explicit, use ansatz, choose normalizaation the finitely many. normalizations, different equivalences here eqg to complex, note eql to real. two possibilities here unclear, only numerically. other cases symmetric and eta2 complex, so unique

3 Lattès maps and Thurston maps

Well-known results on rational maps and Thurston maps of Lattès type (2, 2, 2, 2) are explained; the presentation emphasizes the affine lift [2, 29] over the pullback map used in [6, 7].

3.1 Rational Lattès maps

Consider a lattice $\Lambda = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \xi \subset \mathbb{C}$ with $\mathrm{Im}(\xi) > 0$, and an affine map $L(w) = \eta \cdot w + \kappa$ with $\eta \cdot \Lambda \subset \Lambda$, which covers a self-map of the torus $\mathbb{C}/\Lambda$. From a branched cover $\mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$, we obtain a postcritically finite rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $D = |\eta|^2$, a Lattès map. Under the symmetry $w \mapsto -w$ in particular, with $\kappa \in \Lambda/2$, f has four postcritical points and all critical points are non-degenerate and not postcritical, which is symbolized by the orbifold type (2, 2, 2, 2). There is an even Weierstraß function $\wp : \mathbb{C} \to \widehat{\mathbb{C}}$ with $f \circ \wp = \wp \circ L$. Probably the best known examples are flexible or integral Lattès maps, which exist when D is a square: for $\eta = \sqrt{D}$ the generator ξ is arbitrary, giving a one-parameter family of quasi-conformally conjugate rational maps, which have invariant line fields. We shall see that already in degree $D = 2$, Lattès maps have many interesting properties. See [18, 20, 2] for other properties and alternative characterizations.

cover is branched on $\Lambda/2$, maps to postcritical set P of f. $f^{-1}(P) = \Omega \cup P$

From now on, we assume the degree is $|\eta|^2 = 2$. Since $\eta \cdot \Lambda \subset \Lambda$, there are integers a, b, c, d such that

$$\eta \cdot 1 = a + c\xi \qquad \eta \cdot \xi = b + d\xi \; . \tag{4}$$

Due to the symmetry $w \mapsto -w$, f is covered by another map as well, where η is replaced with $-\eta$, but η^2 characterizes f. Different choices of κ may give equivalent maps, cf. Proposition 3.1.2. A change of ξ means choosing a different fundamental cell in the same lattice Λ, and the matrix A with components a, b, c, d will be conjugated with a matrix $S \in SL_2(\mathbb{Z})$. In the quadratic rational case, we always have $bc \neq 0$, and $\mathrm{Im}(\eta) = \mathrm{Im}(c\,\xi) \neq 0$. Now (4) gives the following relations,

$$\eta^2 - (a + d)\eta + 2 = 0 \qquad c\,\xi^2 + (a - d)\,\xi - b = 0 \; , \tag{5}$$

and the determinant is $ad - bc = 2$. In particular, we have $|a + d| \leq 2$ and there are only finitely many values possible for η^2. Moreover, there are only three branch portraits of type (2, 2, 2, 2) in degree two. Grouping Möbius conjugate maps and complex conjugate maps together, it turns out there are four cases of quadratic rational maps of type (2, 2, 2, 2); an overview is given in Table 2.

3.2 The Thurston characterization

define Thurston map $\widetilde{g}$, postcritical set P, Thurston's combinatorial equivalence $\sim$

(question of rational map, reference to obstructions)

application to constructing rational maps, and matings in particular. A quadratic Thurston map $\widetilde{g}$ of type (2, 2, 2, 2) can be constructed as follows: take the lattice $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$, fix an even cover $\mathbb{R}^2/\mathbb{Z}^2 \to \widehat{\mathbb{C}}$, choose an affine map $L(\vec{x}) = A\vec{x} + \vec{\kappa}$

Quadratic rational Lattès maps f of type (2, 2, 2, 2)	
Branch portrait a) and b) $\Rightarrow\ \bullet\ \searrow$ $\qquad\bullet\ \to\ \bullet\ \uparrow$ $\Rightarrow\ \bullet\ \nearrow$	c) $\Rightarrow\ \bullet\ \to\ \bullet\ \uparrow$; $\Rightarrow\ \bullet\ \to\ \bullet\ \uparrow$ d) $\Rightarrow\ \bullet\ \to\ \bullet$; $\updownarrow$; $\Rightarrow\ \bullet\ \to\ \bullet$
a) $u = 1 \pm \sqrt{2}$, $\eta^2 = -2$, $\kappa = 0$ b) $c = \pm\mathrm{i}$, $\eta^2 = \mp 2\mathrm{i}$, $\kappa = 0$	c) $c = \frac{1\pm\sqrt{7}\mathrm{i}}{2}$, $\eta^2 = \frac{-3\pm\sqrt{7}\mathrm{i}}{2}$, $\kappa = 0$ d) $c = \frac{1\pm\sqrt{7}\mathrm{i}}{4}$, $\eta^2 = \frac{-3\mp\sqrt{7}\mathrm{i}}{2}$, $\kappa = 1/2$
a) is symmetric under complex conjugation, b) under inversion.	Both c) and d) are symmetric under inversion.
The Thurston pullback σ_f, the pullback of simple closed curves, and the virtual endomorphism $\Phi_f : H \to G$ of the pure mapping class group are of finite order.	All of these relations have no periodic orbits except for the unique fixed points of σ_f and Φ_f. (See Section 6.1 and [10].)
In an even map cover, σ_f projects to an isomorphism of moduli space, and a moduli space map k exists there.	The correspondence on moduli space is not reduced in a covering space [10].
Some iterate of f is a flexible Lattès map.	No iterate of f is flexible.

Quadratic Thurston maps $\widetilde{g}$ of Lattès type (2, 2, 2, 2)	
The trace of A is even. All values of $\vec{\kappa}$ are equivalent.	The trace is odd. Changing $\vec{\kappa}$ gives either c) or d).
$\widetilde{g}$ cannot have an obstruction.	$\widetilde{g}$ may be multicurve obstructed.

Table 2: Up to conjugation with a Möbius transformation or complex conjugation, there are four cases of quadratic rational maps of type (2, 2, 2, 2); see Sections 3.4 and 3.5 for concrete formulas. Thurston maps of type (2, 2, 2, 2) are discussed in Sections 3.2 and 3.3.

with $A \in \mathbb{Z}^{2\times 2}$ of determinant 2 and $\vec{\kappa} \in \mathbb{Z}^2/2$, and define $\widetilde{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that it is covered by L. The notation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad : \quad \begin{pmatrix} d \\ -c \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} -b \\ a \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{6}$$

is compatible with (4) and (5); so η is an eigenvalue of A and ξ corresponds to the second base vector, if it is not real. Conversely, every quadratic Thurston map $\widetilde{g}$ of type (2, 2, 2, 2) is equivalent to a map covered by an affine map in this way. See [6, 7, 2, 29] for the proof, which is based on the intermediate lift to a torus and on the identification of a homology group with $\mathbb{Z}^2$, or on the notion of a universal orbifold cover.

Proposition 3.1 ()
1. *A quadratic Thurston map $\widetilde{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is of type (2, 2, 2, 2), if it has four postcritical points and no critical point is postcritical. Assuming that there are no additional marked points, it is combinatorially equivalent to a map covered by a real affine map $L(\vec{x}) = A\vec{x} + \vec{\kappa}$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$ modulo $\vec{x} \mapsto -\vec{x}$.*
2. *The possible branch portraits according to Table 2 are related to the parity of the*

trace $t = a + d$*:*

- *When* t *is even, the branch portrait is a)-b), and all values of the translation* $\vec{\kappa} \in \mathbb{Z}^2/2$ *give conjugate affine maps* L*.*
- *When* t *is odd, changing* $\vec{\kappa}$ *gives two different affine conjugacy classes, which correspond to the branch portraits of case c) and d), respectively.*

3\. *Suppose the Thurston maps* $\widetilde{g}$ *and* $\widetilde{g}'$ *are covered by the affine maps* $L(\vec{x}) = A\vec{x} + \vec{\kappa}$ *and* $L'(\vec{x}) = A'\vec{x} + \vec{\kappa}'$*, respectively. Then* $\widetilde{g}'$ *is combinatorially equivalent to* $\widetilde{g}$*, if and only if* L' *is affine conjugate to* $\pm L$ *modulo* $\mathbb{Z}^2$*, such that* A' *is similar to* $\pm A$ *with a conjugator* $S \in SL_2(\mathbb{Z})$*.*

Proof: 1. See the references given above, and [29] for the case of additional marked points; then a lift is possible unless there are removable Lévy cycles. lift in A.8 in [2]. Note two cases and three kinds of arcs.

2\. The branch portrait is obtained by checking all combinations of parity; a priori there are sixteen combinations, but a few are ruled out by determinant 2. An explicit calculation checks whether two translations in $\mathbb{Z}^2/2$ are equivalent in $\mathbb{Z}^2$ by conjugating with a translation in $\mathbb{Z}^2/2$.

3\. Combinatorial equivalences ψ_0, ψ_1 are covered by maps isotopic to the same affine map; this map sends $\mathbb{Z}^2/2$ to itself, so $S \in SL_2(\mathbb{Z})$. We cannot distinguish between A and $-A$, since the cover identifies $\vec{x}$ and $-\vec{x}$. ■

Theorem 3.2 (Thurston characterization)
Consider a quadratic Thurston map $\widetilde{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *of type* $(2, 2, 2, 2)$*, which is covered by the real affine map* $L(\vec{x}) = A\vec{x} + \vec{\kappa}$ *on* $\mathbb{C}/\mathbb{Z}^2$ *up to isotopy. Then* $\widetilde{g}$ *is combinatorially equivalent to a rational map* f*, if and only if the eigenvalues of* A *are not real, or equivalently, if the trace is* $|t| \le 2$*.*

Proof: If the eigenvalues are not real, determine ξ and η from (4) and (5) with $\mathrm{Im}(\xi) > 0$. Conjugate L with an affine map $\mathbb{R}^2 \to \mathbb{C}$, which is sending the base vectors to 1 and ξ. The new map will be of the form $\eta \cdot w + \kappa$, so it covers a rational Lattès map f. Conversely, any rational f is described by an affine map with non-real eigenvalues according to the previous Section 3.1, and the matrices must be conjugate by Proposition 3.1.3. ■

In practice, the affine lift of a Thurston map g can be obtained as follows: choose a simple closed curve γ through the four postcritical points, and cover $\widehat{\mathbb{C}}$ by $\mathbb{R}^2/\mathbb{Z}^2$ such that the interior of γ is covered by $[0,\, 1/2]^2$. Lift $\gamma' = \widetilde{g}^{-1}(\gamma)$ to $\mathbb{R}^2$. The curves will be $\mathbb{Z}^2$-periodic and isotopic to straight lines through lattice points. Choose a fundamental cell and determine the affine map L sending this parallelogram to $[0,\, 1]^2$. The coefficients of A are read off from (6). Of four adjacent cells, two will give an orientation-preserving map, and these two give $\pm A$.

3.3 The Thurston pullback

dfn Teichmüller space, moduli space, π, and pullback map $\sigma_{\widetilde{g}}$; fixed point gives equivalence to rational map [DH, book2, teich]

In the previous Section 3.2, we have characterized Thurston maps $\widetilde{g}$ of type $(2, 2, 2, 2)$ in terms of the trace $t = a + d$ of an associated matrix A, and it was not necessary to consider the pullback map. Now $\sigma_{\widetilde{g}}$ shall be discussed as well for two reasons: to compare the pullback behavior to maps of type not (2, 2, 2, 2), and because the convergence properties of the Thurston Algorithm for certain matings will be investigated in [5].

Theorem 3.3 (Thurston pullback)
Suppose $\widetilde{g}:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is a quadratic Thurston map of type (2, 2, 2, 2), without additional marked points. Then Teichmüller space $\mathcal{T}$ is identified with the upper halfplane, and the Thurston pullback is given by a Möbius transformation

$$\sigma_{\widetilde{g}}(\tau)=\frac{d\,\tau+b}{c\,\tau+a} \tag{7}$$

with integer coefficients and determinant 2. There are two possible cases:

- *$\sigma_{\widetilde{g}}$ has a unique fixed point $\tau=\xi$ in the upper halfplane $\mathcal{T}$. Then $\widetilde{g}$ is combinatorially equivalent to a rational Lattès map f, which is unique up to Möbius conjugation. The fixed point is neutral with multiplier $\rho=2/\eta^2$.*
- *$\sigma_{\widetilde{g}}$ does not have a fixed point in $\mathcal{T}$, there is no equivalent rational map, and the Thurston pullback $\tau_n=\sigma_{\widetilde{g}}^n(\tau_0)$ diverges to the boundary of $\mathcal{T}$.*

Proof: explain pullback of constant Beltrami coefficient This gives (7), where a, b, c, d are the coefficients of the matrix A. Now $\sigma_{\widetilde{g}}(\tau)=\tau$ yields the same equation as (5) for ξ, so a fixed point τ in the upper halfplane exists, if and only if there is a ξ with $\mathrm{Im}(\xi)>0$. A short computation gives $\sigma'_{\widetilde{g}}(\xi)=2/\eta^2$. ■

Remark 3.4 (Multicurve obstructions)
When $\widetilde{g}$ is not of type (2, 2, 2, 2), the Thurston pullback is non-uniformly contracting, so a fixed point is unique and globally attracting. An obstructing multicurve Γ for a Thurston map $\widetilde{g}$ has certain properties under pullback, which imply that in the Riemann surfaces defined by τ_n, corresponding hyperbolic geodesics get shorter and annuli get thicker, which may prevent convergence. When $\widetilde{g}$ is not of type (2, 2, 2, 2), the Thurston Theorem says that $\widetilde{g}$ is equivalent to a rational map, if and only if it is unobstructed. This is not true if $\widetilde{g}$ is of Lattès type (2, 2, 2, 2):

- *When $\widetilde{g}$ is quadratic and the matrix A from the affine lift of $\widetilde{g}$ has trace $|t|\ge 4$, there will be no obstruction, but $\widetilde{g}$ is not equivalent to a rational map either. The pullback is bounded in moduli space, but diverges to the boundary in Teichmüller space.*
- *Only for $|t|=3$ there is an obstruction, and the iteration diverges to the boundary in moduli space as well, since the obstruction is pinching. According to Selinger [28], invariant essential curves are related to integer eigenvectors of A, and so determinant 2 requires trace ± 3. Note that an obstructed map will be of case c) or d) according to Proposition 3.1.2. Alternatively, the core arc argument shows that maps of case a)-b) are unobstructed, since an invariant essential curve would contradict the branch portrait.*
- *Quadratic rational maps are always unobstructed, but when the degree is a square, there exists a family of flexible Lattès maps, which have a non-pinching obstruction in fact.*

3.4 The rational maps of cases a) and b)

Now we shall determine the quadratic rational Lattès maps f of type (2, 2, 2, 2) explicitly, by observing the branch portraits visualized in Table 2. See also [18, 20]. The pullback correspondence on moduli space is discussed as well. The affine maps from the previous Sections 3.2–3.3 will not be used to obtain f; but to see which rational map corresponds to which value of η, either the relation $\rho = 2/\eta^2$ can be employed, or the fact that a fixed point of f has multiplier $\pm\eta$ if it is not postcritical, and multiplier η^2 if it is.

The branch portrait of cases a) and b) is the same: both critical values are mapped to the same prefixed point. Assume that the critical points are 0 and ∞, so that f is even, and put the postcritical fixed point at 1, so $f(\pm1) = 1$. Denoting the critical values by $f(0) = u$ and $f(\infty) = -u$, we have functions of the form F_u with

$$F_u(z) = -u\frac{z^2 + \frac{1+u}{1-u}}{z^2 - \frac{1+u}{1-u}} \qquad F_u^{-1}(z) = \sqrt{\frac{1+u}{1-u} \cdot \frac{z-u}{z+u}}\ . \tag{8}$$

These functions will be used both for the Thurston pullback, where the parameter u varies, and to determine specific values of u representing Lattès maps. Now the condition $F_u(\pm u) = -1$ gives $(u^2+1)(u^2-2u-1) = 0$.

Case a) is given by $u = 1 \pm \sqrt{2}$, so $F_{1\pm\sqrt{2}}(z) = -(1 \pm \sqrt{2})\frac{z^2-(1\pm\sqrt{2})}{z^2+(1\pm\sqrt{2})}$. The two maps are real in this normalization, not symmetric under inversion, but they are transformed into each other by the inversion in fact; so they belong to the same combinatorial equivalence class, when critical points are not marked. We have $\eta^2 = F'_{1\pm\sqrt{2}}(1) = -2$ and $\kappa = 0$.

Case b) shall denote the maps with $u = \pm\mathrm{i}$, $F_{\pm\mathrm{i}}(z) = \mp\mathrm{i}\,\frac{z^2\pm\mathrm{i}}{z^2\mp\mathrm{i}}$. The two maps are complex conjugate to each other, and each is invariant under conjugation with the inversion $z \mapsto 1/z$, so it can be written in the form $f_{\pm\mathrm{i}}(z) = \frac{z^2\pm\mathrm{i}}{1\pm\mathrm{i}z^2}$ according to (10) as well. Computing $F'_u(1) = (1-u^2)/u$ gives $\eta^2 = -2\mathrm{i}$ for f_i and $\eta^2 = 2\mathrm{i}$ for $f_{-\mathrm{i}}$, and we may assume $\kappa = 0$.

In this normalization at 0 and ∞, moduli space is given by $u \in \widehat{\mathbb{C}} \setminus \{0,\ \infty,\ -1,\ 1\}$. In case a) or b), the Thurston pullback σ_f of f defines a correspondence on moduli space, such that u is pulled back to u'. Now u determines F_u by its critical values, and u' satisfies $F_u(\pm u') = -1$ since $F_u(-1) = 1$. This example has the special property, that the correspondence is reducible: (8) gives

$$u' = \pm\frac{1+u}{1-u}\ , \qquad \textbf{a)}\ u' = -\frac{1+u}{1-u} \qquad \textbf{b)}\ u' = +\frac{1+u}{1-u}\ . \tag{9}$$

Here the sign is determined from the known values of u at the fixed point of σ_f; it is the same sign globally by analytic continuation. The multiplier ρ of the Thurston pullback is computed either from (9) or from the general relation $\rho = 2/\eta^2$, which gives $\rho = -1$ in case a), and $\rho = \pm\mathrm{i}$ for $F_{\pm\mathrm{i}} = f_{\pm\mathrm{i}}$, $\eta^2 = \mp2\mathrm{i}$ of case b).

Remark 3.5 (Thurston pullback map)
1. The Thurston pullback map is of finite order, σ_f^2 or σ_f^4 is the identity. This can be seen either from the fact that it is a Möbius transformation of the upper halfplane with a rationally neutral fixed point, or by noting that at its fixed point, σ_f is analytically conjugate to a branch of the correspondence (9) on moduli space,

and employing analytic continuation. Note that $\pi : \mathcal{T} \to \mathcal{M}$ is an infinite–to–one cover semiconjugating the Möbius transformation σ_f to the Möbius transformation (9). When g is a Thurston map of type $(2, 2, 2, 2)$ with branch portrait a)-b), the correspondence on moduli space is given by (9) as well, but σ_f will not be of finite order, if it does not have a fixed point.

2. In case a), f^2 is a flexible Lattès map of degree four (see also Section 6.6). Since f is a geometric mating according to Theorem 1.2, the composition is represented by a mating, whose combinatorial equivalence class does not determine a unique Möbius conjugacy class; this observation is due to Pilgrim [18]. Now σ_f can be used to obtain a parametrization for the flexible family: with $u' = -(1+u)/(1-u)$, define $I_u = F_u \circ F_{u'}$.

3. The correspondence on moduli space is reducible only in the "even" normalization with marked critical points, which is a double cover of ordinary moduli space in fact, see [10].

4. Since the maps $f_{\pm\mathrm{i}}$ are symmetric under inversion, we may look at the pullback map restricted to symmetric maps. It turns out that this map is actually constant, since $f_c(\pm\mathrm{i}) = -1$ for all parameters c. Locally there are two invariant manifolds, one with even maps and multiplier $\mp\mathrm{i}$, one with symmetric maps and multiplier 0; the first step of the pullback lands on the even one, and symmetric maps land on the fixed point. We shall see in [5] that slow mating converges for the self-mating of $z^2 + \gamma_M(1/4)$, which is related to the eigenvalue of the invariant manifold being 0 instead of neutral (see also Section 6.3).

3.5 The rational maps of cases c) and d)

Consider the following one-parameter families of quadratic rational maps, with $c \neq \pm 1$, $u \neq 0,\, 1$, or $u \neq \pm 1$. Again they are normalized with critical points 0 and ∞, and f_c is symmetric with respect to conjugation by the inversion $z \mapsto 1/z$:

$$f_c(z) = \frac{z^2 + c}{1 + cz^2} \qquad h_u(z) = \frac{z^2 - \frac{2u}{u+1}}{z^2 - \frac{2}{u+1}} \qquad H_u(z) = \frac{z^2 - \frac{u+1}{2}}{z^2 - \frac{u+1}{2u}} \tag{10}$$

Case c) is a Lattès map with disjoint critical orbits, such that both critical values are mapped to fixed points. Now h_u according to (10) satisfies $\infty \Rightarrow 1 \to -1 \uparrow$, and $0 \Rightarrow u \to -u \uparrow$ requires $h_u(\pm u) = -u$ or $u(u-1)(u^2 + 3u + 4) = 0$; here $u = 1$ is excluded and $u = 0$ has a different branch portrait. So $u = (-3 \pm \sqrt{7}\mathrm{i})/2$ gives two complex conjugate Lattès maps h_u . It turns out tht these are rescaled to symmetric maps f_c with $c = (1 \pm \sqrt{7}\mathrm{i})/2$. We have $\eta^2 = (-3 \pm \sqrt{7}\mathrm{i})/2$ and $\kappa = 0$.

The Thurston pullback induces a correspondence on moduli space; $h_u(\pm u') = -u$ gives

$$h_u^{-1}(z) = \sqrt{\frac{2}{u+1} \cdot \frac{z-u}{z-1}}\ , \qquad u' = h_u^{-1}(-u) = \frac{2\sqrt{u}}{u+1}\ . \tag{11}$$

The irrationally neutral fixed points at $u = (-3 \pm \sqrt{7}\mathrm{i})/2$ have the multipliers $\rho = (-3 \mp \sqrt{7}\mathrm{i})/4$. There is a superattracting fixed point at $u = 1$ indicating a possible pinching obstruction; $u = 0$ is not attracting.

Case d) denotes a Lattès map with a postcritical 2-cycle. It shall have the following branch portrait: $0 \Rightarrow u \to -1 \leftrightarrow -u \leftarrow 1 \Leftarrow \infty$. This is provided by H_u

if u satisfies $H_u(\pm u) = -1$, or $(u-1)(4u^2+3u+1) = 0$. So there are two complex conjugate maps H_u with $u = (-3 \pm \sqrt{7}\mathrm{i})/8$. Again, they are rescaled to symmetric maps f_c with $c = (1 \pm \sqrt{7}\mathrm{i})/4$. Then $\pm\eta = f_c'(1)$ shows $\eta^2 = (-3 \mp \sqrt{7}\mathrm{i})/2$ for the affine lift, and $\kappa = 1/2$ gives the correct branch portrait.

Now consider the Thurston pullback with $\pi(\tau) = u$ and $\pi(\sigma_f(\tau)) = u'$ in the even normalization H_u. The correspondence on moduli space is determined from $H_u(\pm u') = -1$ as

$$H_u^{-1}(z) = \sqrt{\frac{u+1}{2u} \cdot \frac{z-u}{z-1}} \ , \qquad u' = H_u^{-1}(-1) = \frac{u+1}{2\sqrt{u}} \ . \tag{12}$$

At the parameters $u = (-3 \pm \sqrt{7}\mathrm{i})/8$, a branch has a neutral fixed point with the multiplier $\rho = (-3 \pm \sqrt{7}\mathrm{i})/4 = 2/\eta^2$. Note that $\rho^2 + \frac{3}{2}\rho + 1 = 0$ shows that the fixed point is irrationally neutral; I do not know whether it is Brjuno, but a local branch of (12) will be linearizable anyway, because it is conjugate to the Möbius transformation σ_f. The pullback relation has a superattracting fixed point $u = 1$ in addition, which does not correspond to a rational map, but indicates that Thurston maps with branch portrait d) may have a pinching obstruction. Moreover, this fixed point is related to the convergence of arithmetic-geometric means [21]. In both cases c) and d) the Teichmüller space and moduli space contain another invariant manifold corresponding to symmetric maps. The pullback relation reads

$$\textbf{c)} \quad c' = \sqrt{-\frac{2c}{c^2+1}} \ , \qquad \textbf{d)} \quad c' = \sqrt{-\frac{c^2+1}{2c}} \ . \tag{13}$$

These pullback relations are locally conjugate to (11) and (12), respectively, via $u = c^2$. So they have the same neutral multiplier ρ at corresponding fixed points, in contrast to case b) according to Remark 3.5.4. — Note that the affine lifts of cases c) and d) have the same η^2 but differ in the translation κ; the rational maps are related, e.g., as follows: if f_c is of case c), then $f_{1/c}$ is of case d), and $f_c^2 = f_{1/c}^2$.

4 Lattès maps as matings

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4.1 Polynomial dynamics and combinatorics

P, K, rays, landing, persistence behind root [17, 27]

in next subsection mention phi(K), say fixed either 0ray or length 2 or length 4, need to bound denominator/ray period

In Sections 4.2 and 6.5, we shall need special results on periodic cycles, to find or to exclude certain types of ray connections, and to characterize essential matings with specific ramification portraits. Item 1 is proved by counting endpoints of Hubbard trees, and items 2 and 3 mean that the dynamics for a rotation number with high denominator is rigid with respect to small changes. See also Proposition 4.1.c in [9].

Lemma 4.1 (Combinatorics of quadratic polynomials)
Consider a critically preperiodic polynomial $P(z) = z^2 + p$.

1. *Suppose* p *has preperiod* k, *and the corresponding periodic cycle of* $P^k(p)$ *persists from the root* $p' \prec p$. *Then* $k \geq 2$, *and* $k = 2$ *occurs only when* p *is real.*

2. *Suppose* p *has preperiod* k *and it belongs to a limb of denominator* r. *The periodic cycle of* $P^k(p)$ *shall have the same angles as the* α*-fixed point of another limb of the same denominator* r:
a) If $k = 1$, *then* $r = 3$.
b) If $k = 2$ *and the two limbs are conjugate, then* $r = 3$ *or* $r = 4$.

3. *Suppose* p, $\widetilde{p}$ *are parameters of preperiod* 1 *and belong to limbs of denominators* r, $\widetilde{r}$. *Now* $P(p)$ *shall have an angle of* $\alpha_{\widetilde{p}}$ *and* $\widetilde{P}(\widetilde{p})$ *shall have an angle of* α_p. *If* $\widetilde{r} < r$, *then* $\widetilde{r} = 2$ *and* $r = 3$.

Recall that for each hyperbolic component with root $p' \neq 1/4$, there is an associated cycle of primitive or satellite type, whose rays persist for all parameters p$\succeq p'$. Conversely, if a periodic cycle of P does not consist of endpoints, there will be a corresponding root $p' \preceq p$. In particular, when p belongs to the limb of rotation number s/r, the fixed point α_p has an r-cycle of dynamic rays.

Proof: 1. First, assume that p' is the root of a limb with rotation number s/r. Then $P^k(p) = \alpha_p$ requires $k \geq r$, so $k = 1$ is excluded, and $k = 2$ only for $r = 2$ and the real parameter $p = \gamma_M(5/12) = \gamma_M(7/12)$. Second, assume the periodic cycle is $z_1, \ldots, z_m$ with $m \geq 3$ and the characteristic point z_1 separating p from the other points in the cycle. T is the Hubbard tree of P and $T' \subset T$ the connected hull of the m-cycle. Then z_1 and z_2 are endpoints of T', and 0 is an inner point. p is behind z_1 and $P(p)$ behind z_2, i.e., z_2 is separating $P(p)$ from 0. If $k = 1$, then $P(p)$ is a periodic point z_j behind z_2, which contradicts z_2 being an endpoint of T'. If $k = 2$, then z_3 is not an endpoint of T', because $P^2(p)$ would be a periodic point behind it; noting that P is injective on $[z_1, p]$ and on $[z_2, P(p)]$, and an arc before z_2 would be mapped before z_3. So T' has only two endpoints, and $P^2(p) \in T'$ implies that T has two endpoints as well, so p is real.

2a) For $r = 2$, there is no other limb of the same denominator. For $r = 3$, $p = \gamma_M(\pm 3/14)$ belongs to the limb with rotation number $\pm 1/3$, and it is mapped to the angle $\pm 3/7$, which belongs to α of the conjugate limb. For $r \geq 4$ we shall obtain a contradiction: Denote the sectors at α_p by $W_1, \ldots, W_r$ in the order of the orbit of p, with $p \in W_1$ and $0 \in W_r$. The periodic r-cycle of $P(p)$ shall be labeled such that it has corresponding indices, so $z_2 = P(p) \in W_2, \ldots, z_r = P^{r-1}(p) \in W_r$, and $z_1 = P^r(p)$. Now z_1 is the periodic preimage of z_2, so $z_1 = -p$ is behind $-\alpha_p$ and belongs to W_r. The periodic points are endpoints by item 1, and we are interested in the cyclic order of their angles θ_j. Since θ_r and θ_1 are the only angles in W_r, the rotation number must be $\pm 1/r$. Compare these angles to the original sectors: we have removed position 1 and added a new position 1 next to position r. If $r \geq 4$, there are at least two neighboring positions left unchanged, so the rotation number was ± 1 in the limb of p already. This contradicts the hypothesis and item 1.

2b) For $r = 3$ or $r = 4$ we have $p = \gamma_M(\pm 5/28)$ and $p = \gamma_M(\pm 7/60)$, respectively. For $r = 5$ and $r = 6$, no solution is found. It remains to obtain a contradiction for $r \geq 7$: We have $p \in W_1$, $P(p) \in W_2$, $P^2(p) = z_3 \in W_3, \ldots, P^{r-1}(p) = z_r \in W_r$, $P^r(p) = z_1$, and $P^{r+1}(p) = z_2 = -P(p) \in W_r$. Now z_1 is mapped into W_r, so $z_1 \in W_{r-1}$ or $z_1 \in W_r$.

Case 1: $z_1 \in W_{r-1}$ and in W_r we have, say, the cyclic order z_r before z_2. Then the new rotation number is s'/r with $s' = (r+1)/2$, so the old one was s/r with $s = (r-1)/2$. It turns out that compared to the order of the original sectors, two neighboring positions are swapped two times; position 1 is swapped with $r-1$, and position r swapped with 2. But there are other positions jumping over two neighbors, so the rotation number could not have changed.
Case 2: $z_1 \in W_r$ and in W_r we have s' steps from z_r to z_1 and from there to z_2 as well. So without restriction assume $s' = 1$. Then z_{r-1} comes directly before z_r regarding the cyclic order of angles. Since we have no periodic points in W_1 and W_2, only these could be between W_{r-1} and W_r, so the old number of steps s was 1, 2, or 3. This contradicts $s + s' = r$.

3. For $r = 3$ and $\widetilde{r} = 2$, we have $p = \gamma_M(\pm 1/6)$ and $\widetilde{p} = \gamma_M(\mp 5/14)$. So suppose p has rotation number s/r with $r \ge 4$. We may assume $s/r < 1/2$. Denoting the sectors at α_p by $W_1, \dots, W_r$ again, the periodic points are in $W_2, \dots, W_r$: the latter sector is the first one mapped back to W_2, so $\widetilde{r} = r-1$. There are $2s-1$ steps from z_r to its image z_2 and s steps from z_{r-1} to z_r. So $s = 1$, α_p has rotation number $1/r$ and $\alpha_{\widetilde{p}}$ has $1/(r-1)$. The possible angles of the endpoint $\widetilde{p}$ are determined from the inequality $\frac{1}{2^{r-1}-1} < \frac{5}{2(2^r-1)} < \frac{7}{2(2^r-1)} < \frac{2}{2^{r-1}-1}$. However, doubling the two angles in the middle does not give an angle of α_p, which is of the form $\frac{2^j}{2^r-1}$. ■

4.2 Lattès maps of type (2, 2, 2, 2) as matings

Up to inversion and complex conjugation, we have four rational maps f and nine matings g to consider, see Table 1. According to [18], Shishikura has found seven of these matings and determined, which formal mating g corresponds to which rational function f. Interchanging P and Q conjugates the mating with an inversion, and reflection of both angles means complex conjugation of P and Q and of the rational map. Altogether we have thirty matings for eight rational maps up to linear conjugation, or seven rational maps up to Möbius conjugation.

Theorem 4.2 (Possible Lattès matings, following Shishikura)
There are thirty formal matings $g = P \sqcup Q$ of quadratic polynomials, such that the essential mating $\widetilde{g}$ has orbifold type (2, 2, 2, 2), and the parameters p and q are not in conjugate limbs of the Mandelbrot set $\mathcal{M}$. Up to complex conjugation and interchanging P and Q, these matings are represented by the nine matings given in Table 1.

The nine kinds of formal matings are obtained below, and the corresponding rational maps are identified as combinatorial matings in Section 5 from the Shishikura Algorithm [18]; this completes the proof of the Rees–Shishikura–Tan Theorem 1.1 for orbifold type (2, 2, 2, 2). To prove there are only nine types up to Möbius transformation and complex conjugation, we shall employ the following ideas:

- If $\widetilde{g}$ has a postcritical fixed point, a postcritical point of g must belong to a fixed ray-equivalence class. By an observation of Sharland [30], a ray-equivalence class fixed by g must contain a fixed point of P or Q. See Proposition 3.1 in [8] for a more detailed description of rational ray-equivalence classes.

- The ray-equivalence class of β is a single ray, but the class of α provides more possibilities. To build longer ray connections, use rays from different cycles landing together at periodic points, that persist from primitive hyperbolic components before the current parameters. In principle these connections can be arbitrarily long, but when a ray-equivalence class contains an α-fixed point, there will be no primitive hyperbolic component of the same ray period in that limb.

- For a preperiodic parameter of low preperiod k in a limb of high ray period r, the corresponding periodic cycle will follow the rotation for several steps; in certain situations, this places a restriction on r. Specific results were obtained in Lemma 4.1 from polynomial combinatorics.

We shall frequently speak of rays with angle θ connecting $\mathcal{K}_p$ and $\mathcal{K}_{\overline{q}}$; this gives an accurate description of the combinatorics without taking complex conjugate angles all the time, but geometrically it means that the θ-ray of $\mathcal{K}_p$ is joined with the ray of angle $-\theta$ at $\mathcal{K}_q$.

Proof of uniqueness for the branch portrait of cases a)-b): Since the essential mating maps both critical values to the same prefixed point, $P^2(p)$ and $\overline{Q}^2(\overline{q})$ must belong to the same ray-equivalence class, which is fixed by the formal mating g. If this is the 0-ray, we have $p = q = \gamma_M(\pm 1/4)$ since $p = \overline{q}$ is excluded. Otherwise this class contains an α-fixed point of P or $\overline{Q}$; by Möbius conjugation we may assume it to be α_p, as the branch portrait is symmetric.

1. Suppose $P^2(p) = \alpha_p$, then p is real by Lemma 4.1.1 since the preperiod is $k = 2$. So $p = \gamma_M(5/12) = \gamma_M(7/12)$ and the only remaining angles of the same denominator are $1/12$ and $11/12$. Taking one of these for $\overline{q}$ is seen to work, since $\overline{q}$ is not in the same limb as p, and $\overline{Q}^2(\overline{q})$ shares an angle with α_p.

2. Now suppose that $P^2(p)$ is connected to α_p. This connection goes through only one primitive cycle of $\overline{Q}$ and $P^2(p)$ is an endpoint of the ray-equivalence class, since there is only one hyperbolic component with the ray period of α_p in the limb of p. Thus $\overline{Q}^2(\overline{q})$ must belong to the same primitive cycle of $\overline{Q}$, and by Lemma 4.1.1 again, $\overline{q}$ is real. So the cycle is real and joins complex conjugate angles; the angle of $P^2(p)$ is complex conjugate to an angle of α_p. Now Lemma 4.1.2b says that p is in a limb of ray period 3 or 4. Each of these limbs has a unique angle with the required denominator, which defines p, and a unique $\overline{q}$ is found to work: it is real and the primitive cycle at $\overline{Q}^2(\overline{q})$ shares angles with both $P^2(p)$ and α_p. This gives $\pm 5/28 \amalg 13/28$ and $\pm 7/60 \amalg 29/60$. ∎

Proof of uniqueness in case c): In the essential mating $\widetilde{g}$, the critical values are mapped to different fixed points; in the formal mating g, $P(p)$ and $\overline{Q}(\overline{q})$ belong to distinct fixed ray-equivalence classes. Up to Möbius conjugation, we have the following possibilities:

1. If both classes contain β-fixed points, so $p = \overline{q} = -2$, we are in conjugate limbs. The classes are not actually distinct and the essential mating is undefined, since the critical values would coincide. The topological mating would be defined on a line segment instead of a sphere. In this case, the formal mating is of type $(2, 2, 2, 2)$ in fact, and it is obstructed with trace ± 3.

2. Suppose $\overline{Q}(\overline{q}) = \beta_{\overline{q}}$, so $\overline{q} = -2$, and $P(p)$ is in the ray-equivalence class of α_p. Preperiod $k = 1$ and Lemma 4.1.1 give $P(p) \neq \alpha_p$. The ray connection from

$P(p)$ to α_p passes through a single periodic point of $\overline{Q}$, and the angle is complex conjugated since the periodic points are real. So $P(p)$ shares its angle with $\alpha_{\overline{p}}$. By Lemma 4.1.2a, we have $p = \gamma_M(\pm 3/14) \in \mathcal{M}_{\pm 1/3}$. The angle $\pm 3/7$ of $P(p)$ is reflected at the Airplane characteristic point in $\mathcal{K}_{\overline{q}}$ to become $\pm 4/7$, which is an external angle of α_p.

3. Suppose $\overline{Q}(\overline{q}) = \beta_{\overline{q}}$, so $\overline{q} = -2$, and $P(p)$ is in the ray-equivalence class of $\alpha_{\overline{q}}$, which consists of the rays with angles $\pm 1/3$. This gives $p = \gamma_M(\pm 1/6)$.

4. Suppose $P(p)$ is connected to α_p and $\overline{Q}(\overline{q})$ is connected to $\alpha_{\overline{q}}$. Since $k = 1$, Lemma 4.1.1 gives $P(p) \neq \alpha_p$ and $\overline{Q}(\overline{q}) \neq \alpha_{\overline{q}}$. So the former ray-equivalence class contains a primitive cycle of $\overline{Q}$, whose period is greater than the ray period of $\alpha_{\overline{q}}$ and the same as the ray period of α_p. But by the same arguments, the ray period of $\alpha_{\overline{q}}$ is greater than that α_p, which is a contradiction.

5. Suppose $P(p)$ is connected to $\alpha_{\overline{q}}$ and $\overline{Q}(\overline{q})$ is connected to α_p. These connections must be direct, since a longer connection would require a primitive hyperbolic component before p but with period exceeding the ray period of that limb, or analogously for the limb of $\overline{q}$. So $P(p)$ shares its angle with $\alpha_{\overline{q}}$ and $\overline{Q}(\overline{q})$ shares its angle with α_p. The ray periods may be equal or different. In the former case, Lemma 4.1.2a gives $p = q = \gamma_M(\pm 3/14)$; $P(p)$ has the angle $\pm 3/7$, which is found at $\alpha_{\overline{q}}$ as well. When the ray periods are different, Lemma 4.1.3 gives $p \in \mathcal{M}_{\pm 1/3}$ and $\overline{q} \in \mathcal{M}_{1/2}$ or vice versa. So $P(p)$ has the angle $1/3$ or $2/3$, yielding $p = \gamma_M(\pm 1/6)$, and $\overline{Q}(\overline{q})$ has $\pm 1/7$, $\pm 2/7$, or $\pm 4/7$, giving $\overline{q} = \gamma_M(\pm 9/14)$ and $q = \gamma_M(\pm 5/14)$. ∎

Proof of uniqueness in case d): In the essential mating, both critical values shall be mapped to the unique 2-cycle. For $\pm 1/6 \coprod \pm 1/6$ this works, because the 2-cycles of P and $\overline{Q}$ have direct ray connections. Suppose we had a different formal mating with a 2-cycle of ray-equivalence classes, which contain $P(p)$ and $\overline{Q}(\overline{q})$, respectively. Without restriction, the 2-cycle of P is of satellite type and forms the symmetry centers of these ray-equivalence classes, and p is in a sublimb of the period-2 component of $\mathcal{M}$. Since the preperiod is $k = 1$, Lemma 4.1.1 shows that p is an endpoint of $\mathcal{M}$ and $P(p)$ is an endpoint of its ray-equivalence class. So $\overline{Q}(\overline{q})$ is an interior point of the other ray-equivalence class; it is of primitive type in contradiction to preperiod $k = 1$. ∎

5 The Shishikura Algorithm

5.1 General remarks

aim to prove

Theorem 5.1 (Existence of Lattès matings, following Shishikura)
According to Theorem 4.2, there are nine kinds of formal matings $g = P \sqcup Q$ of quadratic polynomials, such that the essential mating $\tilde{g}$ has orbifold type $(2, 2, 2, 2)$, and the parameters p and q are not in conjugate limbs of $\mathcal{M}$. In each case, the essential mating $\tilde{g}$ is combinatorially equivalent to a rational map $f \simeq P \coprod Q$, which is given in Table 1 as well.

both proof of matability and identification.
two combinatorial steps

- to model essential find curve and preimage, here not with multicon pseudo eq but with angles on circle: two preper polys, choose one point in each postcrit class, preimage of crv isc rossing, rest maybe connected inside or outside
- lift when (2, 2, 2, 2), affine map as in . . . 2/3, complex eigenvalues give combinatorial mating as in ... 2/3

Remark 5.2 (Laminations)
[35] [25] for matings from laminations
neither g nor tilde g, identified zu tilde g and topo

[18] called so, only length 1

[18] curve is equator, one point in each ray-equivalence class, lifted to curves on torus, preimages correspond to subintervals of angles, no mention of problems with longer ray connections.

in 2 several concepts, julia sets and pseudo and essential, here simplified to angles and connections

5.2 Ray connections of length 1

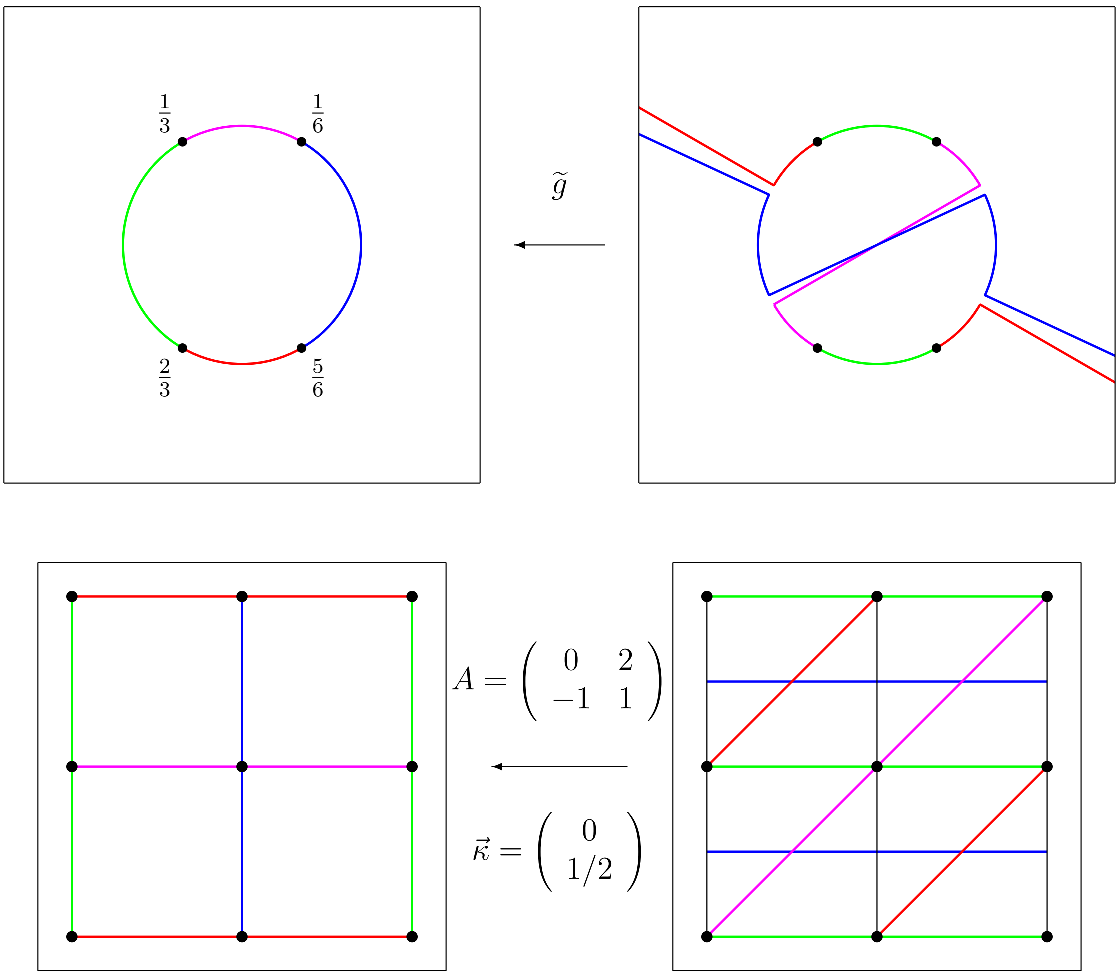

Figure 5: $f \cong 1/6 \coprod 1/6$.

$f \cong 1/6 \coprod 1/6$ yet $f \cong 1/4 \coprod 1/4$

5.3 Ray connections of length 2

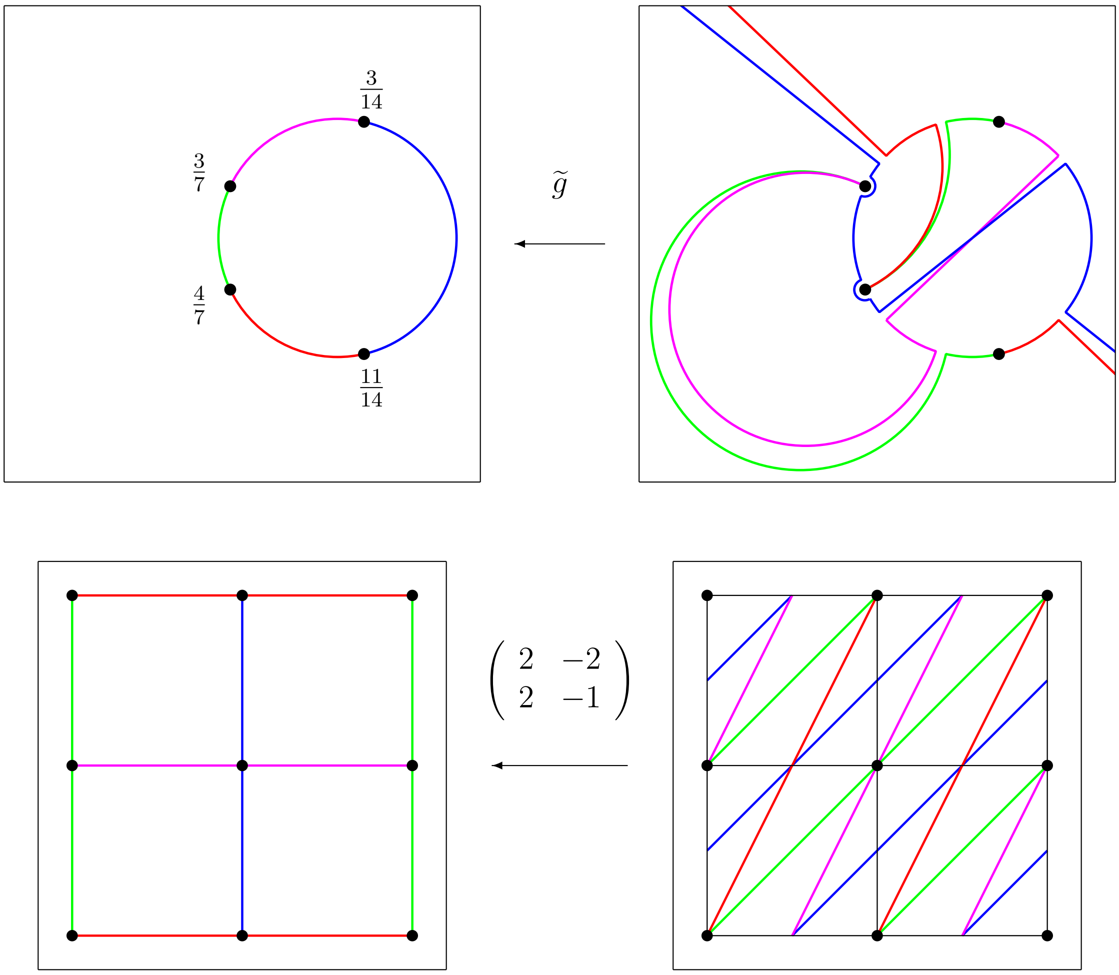

Figure 6: $f \cong 3/14 \amalg 3/14$.

yet $f \simeq 1/12 \amalg 5/12$ $f \cong 3/14 \amalg 3/14$ peter $f \simeq 3/14 \amalg 1/2$ peter $f \simeq 5/6 \amalg 1/2$ $f \cong 1/6 \amalg 5/14$

5.4 The Petersen transformation

$f_c(z) = \frac{z^2+c}{1+cz^2}$. Since $f_{1/c}(z) = 1/f_c(1/z)$,

The Petersen transformation is a semi-conjugation from both f_c and $f_{1/c}$ to the same Chebyshev map [37, 18, 13]. In particular, if $p \in \mathcal{M}$ is postcritically finite and not in the 1/2-limb, then $P \amalg P$ is semi-conjugated to $P \amalg T$ with the Chebyshev polynomial $T(z) = z^2 - 2$. So $\pm 1/4 \amalg \pm 1/4$ are semi-conjugated to the map of type (2, 4, 4) according to Section 6.5, and both $3/14 \amalg 3/14$ and $5/6 \amalg 5/6$ are semi-conjugated to $3/14 \amalg 3/14$.

peter $f \simeq 3/14 \amalg 1/2$

peter $f \simeq 5/6 \amalg 1/2$

5.5 Ray connections of length 4

$f \simeq 5/28 \amalg 13/28$ similarly $f \simeq 7/60 \amalg 29/60$

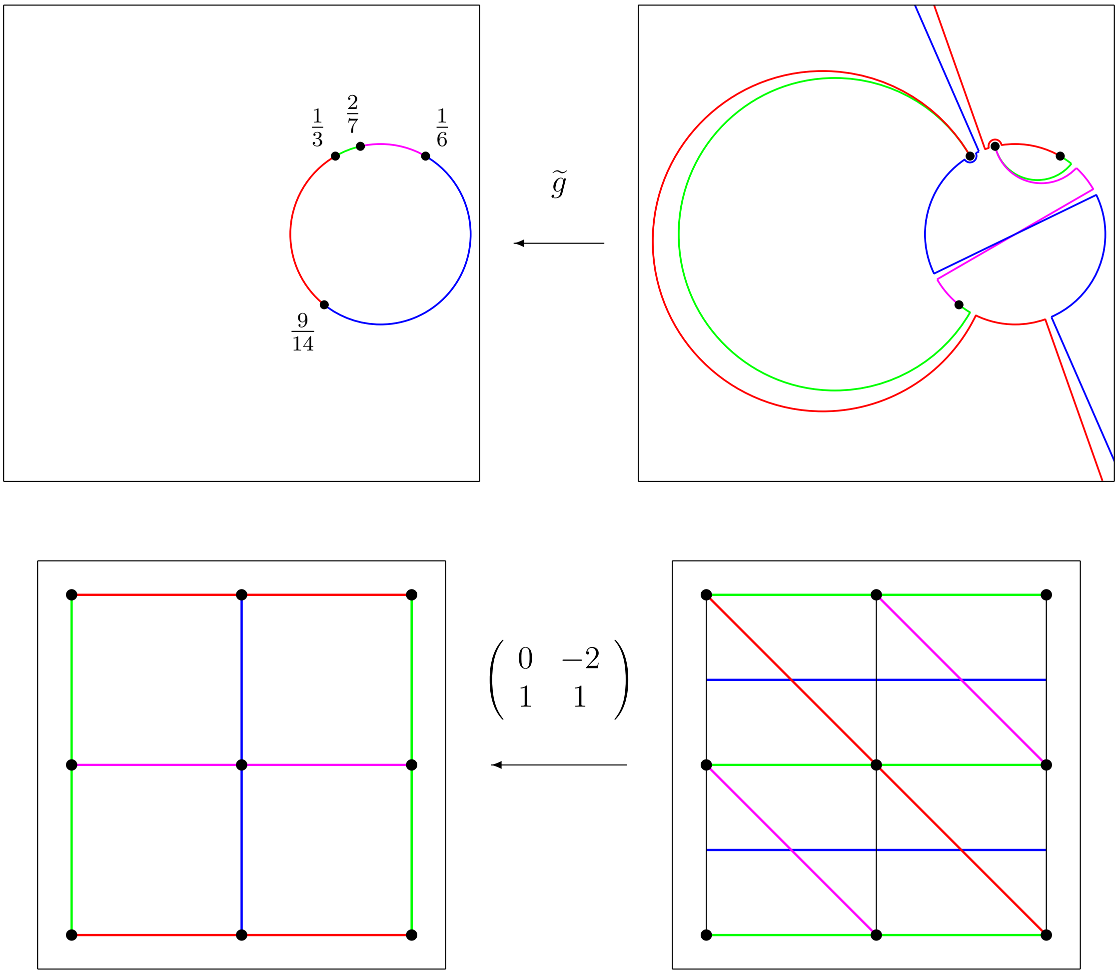

Figure 7: $f \cong 1/6 \coprod 5/14$.

6 Related questions and results

...ref to question choice in Remark 2.3 ...

6.1 Algebraic aspects of Thurston theory for Lattès maps

[10]

curves and endo, maybe reduction, or above already in 3
not hurwitz and msm
twisted lattes

6.2 Anti-matings

A quadratic anti-mating $f \cong P \coprod Q$ is constructed analogously to a mating, such that the formal anti-mating $g = P \sqcap Q$ interchanges the two half-spheres [1, 13]. The quartic Julia sets $\mathcal{K}_{Q \circ P}$ and $\mathcal{K}_{P \circ Q}$ are glued along their boundaries, and the parameters p and q are chosen such that the quartic dynamics is postcritically finite. They are not related to the Mandelbrot set unless $p = q$, which gives a symmetric rational map of the form $f_c(z) = \frac{z^2+c}{1+cz^2}$. Since $f_{1/c}(z) = 1/f_c(1/z)$, we have $f_c \cong P \coprod P$ if and only if $f_{1/c} \cong P \coprod P$, and both are semi-conjugated to $P \coprod T$ by the

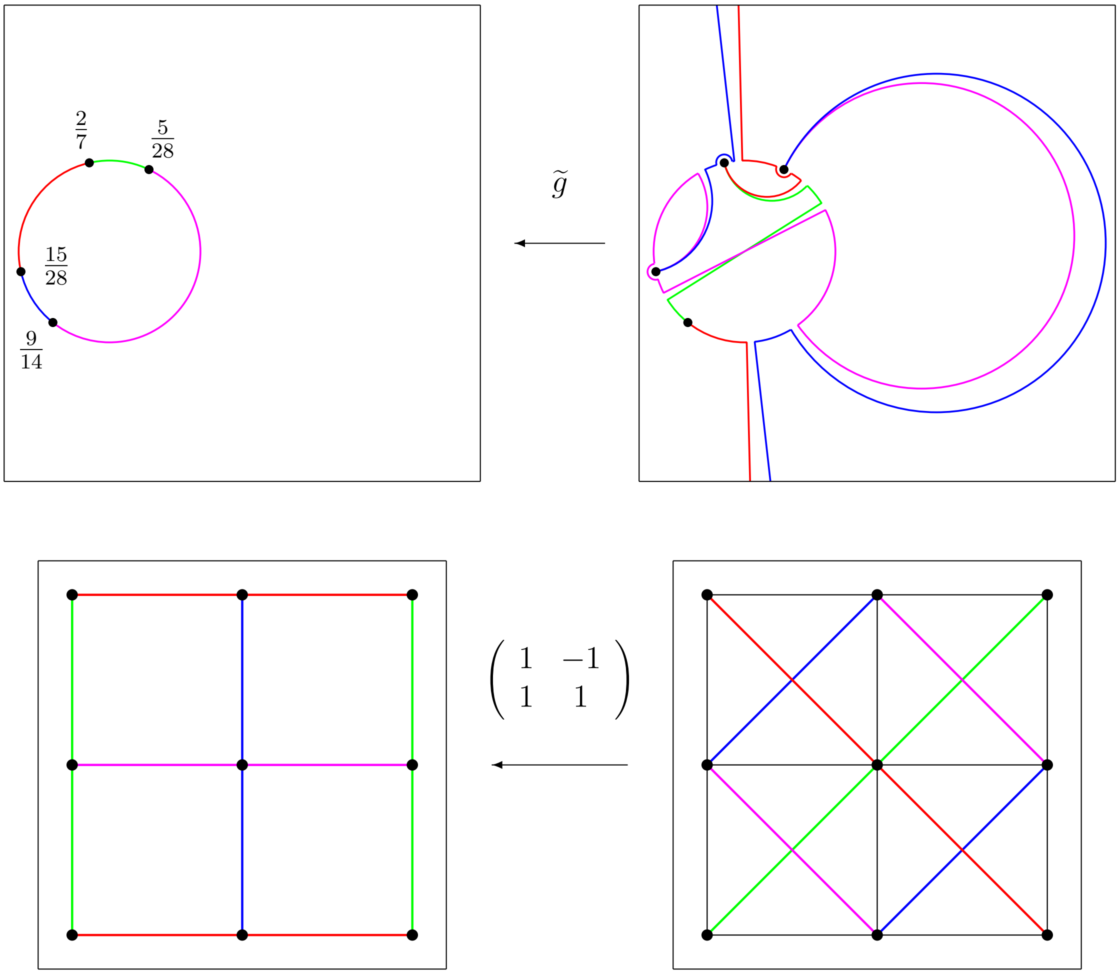

Figure 8: $f \simeq 5/28 \coprod 13/28$.

Petersen transformation (see Section 5.4). Computing $1/c$ for the three kinds of self-mating in Table 1 gives the self-anti-matings

$$1/4 \coprod 1/4 \cong 3/4 \prod 3/4, \quad 1/6 \coprod 1/6 \cong 11/14 \prod 11/14, \quad 3/14 \coprod 3/14 \cong 5/6 \prod 5/6. \quad (14)$$

According to Ahmadi Dastjerdi [1], an essential anti-mating $\widetilde{g}$ is unobstructed, if the three fixed rays of the quartic polynomials land at different points; again, if $\widetilde{g}$ is of type (2, 2, 2, 2), this does not suffice to obtain a rational map, and we need to check the eigenvalues of the associated matrix. For the branch portraits of cases a)-b) and c), it turns out that g must be a self-anti-mating, possibly rotated, and we are done with (14). However, so far I have been unable to show that the only anti-mating of kind d) is $\pm 3/14 \prod \pm 3/14$.

6.3 Convergence of slow mating

[11] otherwise convergent

[4] for distinction (2, 2, 2, 2) not parabolic

[5] here not, spiraling

except 1414, explanation symmetry not normalization–check for other normalization invariant manifold, and petersen

videos

6.4 Matings as Rees captures

[26] [12] general statement, non-convergence even 14v14

shared, here only a

[28] for canonical, [11] for equivalence

6.5 Lattès maps of type (2, 4, 4)

The notion of an orbifold, its type and its universal cover, is explained in [19, 2]. Most types of Thurston maps g or postcritically finite rational maps f have a hyperbolic orbifold, but there are a finite number of types with parabolic orbifold [6, 7]. These maps are covered by affine maps on a cylinder or a torus; the latter are Lattès maps [20, 2]. We have three examples of the former in the quadratic case: $z^{\pm 2}$ is of type $(\infty,\,\infty)$ and z^2-2 of type $(2,\,2,\,\infty)$. The polynomials are trivial matings $z^2 \cong z^2 \coprod z^2$ and $(z^2-2) \simeq (z^2-2) \coprod z^2$, and we have the prototypical anti-mating $z^{-2} \cong z^2 \prod z^2$.

Lattès maps of type (2, 2, 2, 2) have been discussed in the previous sections. There is only one further type in the quadratic case: the branched cover $\mathbb{C} \to \widehat{\mathbb{C}}$ or $\mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ and the map L are symmetric with respect to a quarter rotation, and triangular domains correspond to half-spheres. The rational map f has three postcritical points, including a critical point that is the image of the other critical point, and the orbifold type is (2, 4, 4). Maps of this type are Möbius conjugate to $f(z) = -1 + 2/z^2$ with the branch portrait $0 \Rightarrow \infty \Rightarrow -1 \to 1 \uparrow$. This map does not occur as a formal mating, but as an essential and geometric mating in cases where the formal mating has hyperbolic orbifold:

Theorem 6.1 (Matings of essential type (2, 4, 4))
The rational map $f(z) = -1 + 2/z^2$ of orbifold type (2, 4, 4) is a geometric mating with $f \cong 1/4 \coprod 1/2$, $f \simeq 5/12 \coprod 1/6$, and $f \simeq 13/28 \coprod 3/14$. These are the only representations up to complex conjugation.

Moreover, f is given by the geometric anti-mating $z^2 \prod (z^2+q)$ with $q^3 = -2$ [13].

Proof: We shall use similar arguments as in Section 4.2 and the same notation, labeling critical values of the formal mating $g = P \sqcup Q$ as p and $\overline{q}$. Once the essential mating $\widetilde{g}$ is shown to have the same ramification portrait as f, they will be combinatorially equivalent in fact, since there are only three postcritical points and there is only one rational Möbius conjugacy class. Then the geometric and topological matings are obtained from the Rees–Shishikura Theorem [32]. So we must determine all P and Q, such that $P^2(p)$ and $\overline{Q}(\overline{q})$ belong to the same fixed ray-equivalence class:

1. If this class consists of the 0-ray, we have $\overline{q} = \gamma_M(1/2) = -2 = q$ and thus $p = \gamma_M(\pm 1/4) \in \mathcal{M}_{\pm 1/3}$.

2. Suppose this class contains α_p. If $P^2(p) = \alpha_p$, p must be real according to Lemma 4.1.1, since the preperiod is $k = 2$. So $p = \gamma_M(5/12) = \gamma_M(7/12)$, and $\overline{q} = \gamma_M(\pm 1/6)$ has the property that $\overline{Q}(\overline{q})$ shares the angle $\pm 1/3$ with α_p. Now suppose that there was another example with a longer ray connection from $P^2(p)$ to α_p. This connection must have length two, since there is no primitive hyperbolic component of the same ray period in the limb of p. So there is a unique primitive component before $\overline{q}$, such that the cycle persisting behind it shares angles with both

$P^2(p)$ and α_p. Since there are no other points of $\mathcal{K}_{\overline{q}}$ in the ray-equivalence class of α_p, this primitive cycle must contain $\overline{Q}(\overline{q})$ as well. But this contradicts Lemma 4.1.1 since the preperiod is $k = 1$.

3. Suppose the fixed postcritical class contains $\alpha_{\overline{q}}$, then $\overline{Q}(\overline{q})$ is an endpoint connected to $\alpha_{\overline{q}}$ with length two: the points cannot coincide because the preperiod is $k = 1$, and there can be no primitive component of the required period before $\overline{q}$, which would give a longer ray connection. So $P^2(p)$ belongs to the primitive cycle sharing angles with $\overline{Q}(\overline{q})$ and $\alpha_{\overline{q}}$, and preperiod $k = 2$ implies that p is real according to Lemma 4.1.1. Now the angle of $\overline{Q}(\overline{q})$ is complex conjugate to an angle of $\alpha_{\overline{q}}$ and belongs to α_q in the conjugate limb. By Lemma 4.1.2a, the ray period is 3. With $\overline{q} = \gamma_M(\pm 3/14) \in \mathcal{M}_{\pm 1/3}$, the angle $\pm 3/7$ of $\overline{Q}(\overline{q})$ is connected to $\pm 4/7$ of $\alpha_{\overline{q}}$ at the cycle persisting from the Airplane; the parameter $p = \gamma_M(13/28) = \gamma_M(15/28)$ is the only one of preperiod 2 and period 3 behind the Airplane. ∎

6.6 Algorithms and obstructions in higher degrees

flexible and obstructions both ways

[15, 16]

example of algorithm both ways

2 -1

0 2

is a topological mating of two quartic polynomials, with critical leaves (13, 29) (9, 41) (45, 61) /64 and conj (4, 20) (25, 57) (32, 48) / 64 is obstructed and not rational but topo. Counterexample to Theorem 3.1.1 in [23]

polynomials exist by meyer and Poirier [24]

note infinitely many with trace 4

other abstract example with real eigenvalues, t=d=5, Counterexample to Theorem 3.1.2 in [23] and Theorems 0.1, 1.1 in [31]

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